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ASYMPTOTIC METHOD OF SOLVING NEUTRON-DIFFUSION PROBLEMS

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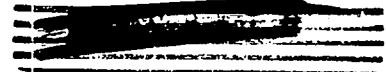
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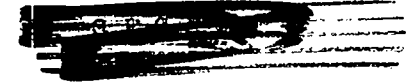
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ABSTRACT

The theory of subcritical systems in two-group approximation is discussed and applied to multiplication and neutron distribution experiments of untamped and tamped systems. On the whole, the agreement between experiment and theory is satisfactory.



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ASYMPTOTIC METHOD OF SOLVING NEUTRON-DIFFUSION PROBLEMS

The method to be discussed has its chief application to subcritical assemblies. The multiplication experiments on 25 cores¹⁾ tamped with WC or U and the neutron distributions in such assemblies²⁾ were the original stimulus to development of the method. As we shall see critical and supercritical are special cases of the treatment.

I Critical System in Asymptotic Approximation

As the asymptotic solution to neutron diffusion involving two velocity groups forms the basis of our calculation, we shall discuss it first.

Assuming spherical symmetry for our core-tamper system, we solve the coupled Boltzmann transport equations for the central point. Serber³⁾ was the first to apply this device to the one-velocity problem. The merit of this approximation is the ease with which the difficult problem of unequal mean free path in core and tamper can be handled.

The two velocity groups, fast and slow, are distinguished by subscripts 1 and 2, respectively. The tamper quantities are characterized by an asterisk (*) and the core symbols are unmarked. The cross sections (in cm^{-1}) are σ , σ_p , σ_{in} , σ_f for the transport, radiative capture, inelastic and fission respectively. We use $N = n v$ as the neutron flux.

The transport equations are then for a core of radius "a" and infinite

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tamper:

$$N_1(0) = \sigma_1(1+f_{11}) \int_0^a e^{-\sigma_1 r} N_1(r) dr + \sigma_2 f_{12} \int_0^a e^{-\sigma_1 r} N_2(r) dr$$

$$+ \sigma_1^* (1+f_{11}^*) \int_a^\infty e^{(\sigma_1^* - \sigma_1) a - \sigma_1^* r} N_1^*(r) dr + \sigma_2^* f_{12}^* \int_a^\infty e^{(\sigma_1^* - \sigma_1) a - \sigma_1^* r} N_2^*(r) dr$$

(1)

$$N_2(0) = \sigma_2(1+f_{22}) \int_0^a e^{-\sigma_2 r} N_2(r) dr + \sigma_1 f_{21} \int_0^a e^{-\sigma_2 r} N_1(r) dr$$

$$+ \sigma_2^* (1+f_{22}^*) \int_a^\infty e^{(\sigma_2^* - \sigma_2) a - \sigma_2^* r} N_2^*(r) dr + \sigma_1^* f_{21}^* \int_a^\infty e^{(\sigma_2^* - \sigma_2) a - \sigma_2^* r} N_1^*(r) dr$$

The angular integration has been already performed in (1). Also, the possible inequality of core and tamper mean free path (σ^{-1}) are in evidence. We have still to define:

$$\sigma_1 f_{11} \equiv (\nu_1 - 1) \sigma_{f_1}^{-\sigma} r_1^{-\sigma} \text{in}; \quad \sigma_1 f_{21} \equiv \nu_2 \sigma_f + \sigma \text{in}$$

(2)

$$\sigma_2 f_{12} \equiv \nu_1 \sigma_{f_2} \quad ; \quad \sigma_2 f_{22} \equiv (\nu_2 - 1) \sigma_{f_2}^{-\sigma} r_2$$

where $\nu_1 \equiv$ number of neutrons emitted into the fast group = $\nu \int_{E_0}^{\infty} \chi_f dE$

and $\nu_2 \equiv \nu \int_0^{E_0} \chi_f dE$; $\nu \equiv$ total number of neutrons per fission.

$\chi_f \equiv$ fission spectrum and $E_0 \equiv$ energy dividing the two velocity groups.

The tamper quantities corresponding to equation (2) are obtained by putting an asterisk everywhere.

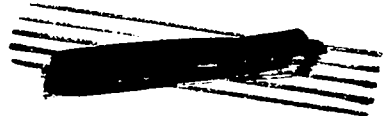
The core asymptotic solution to (1) can be obtained by setting

$$N_1 = A(\sin kr)/kr \text{ and } N_2 = aN_1, \text{ and } a \rightarrow \infty$$

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Then

$$\begin{aligned} \sigma_1 (1+f_{11}^*) K_1^{-1} + a \sigma_2 f_{12}^* K_1 &= 0 \\ \sigma_1 f_{21}^* K_2 + a [\sigma_2 (1+f_{22}^*) K_2^{-1}] &= 0 \end{aligned} \tag{3}$$

where $K_1 \equiv (\arctan k/\sigma_1)/k$
 $K_2 \equiv (\arctan k/\sigma_2)/k$

Equation (3) determines both k and a but not A . k can be either real or pure imaginary. In fact, we write our general core solution as:

$$\begin{aligned} N_1 &= A_1 \frac{\sin k_1 r}{k_1 r} + A_2 \frac{\sinh k_2 r}{k_2 r} \\ N_2 &= a_1 A_1 \frac{\sin k_1 r}{k_1 r} + a_2 A_2 \frac{\sinh k_2 r}{k_2 r} \end{aligned} \tag{4}$$

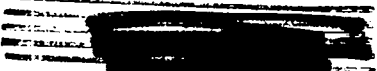
(For the sinh solution in (4), the arctan of K_1 and K_2 of (3) becomes arctanh).

For the taper asymptotic solution, we recall that for the one velocity theory to the characteristic function $e^{-K^* r}/K^*$ belongs the characteristic value $\frac{\operatorname{arctanh} K^*/\sigma_1^*}{K^*}$. Inserting $N_1^* = A^* e^{-K^* r}/K^*$ and $N_2^* = a^* N_1^*$ for $a \rightarrow 0$ in (1), we get

$$\begin{aligned} \sigma_1^* (1+f_{11}^*) K_1^* - 1 + a^* \sigma_2^* f_{12}^* K_1^* &= 0 \\ \sigma_1^* f_{21}^* K_2^* + a^* [\sigma_2^* (1+f_{22}^*) K_2^* - 1] &= 0 \end{aligned} \tag{5}$$

where $K_1^* \equiv (\operatorname{arctanh} K^*/\sigma_1^*)/K^*$; $K_2^* \equiv (\operatorname{arctanh} K^*/\sigma_2^*)/K^*$. Again (5) determines K^* and a^* but not A^* . The general taper asymptotic solution is then

$$\begin{aligned} N_1^* &= A_1^* \frac{e^{-K_1^* r}}{K_1^* r} + A_2^* \frac{e^{-K_2^* r}}{K_2^* r} \\ N_2^* &= a_1^* A_1^* \frac{e^{-K_1^* r}}{K_1^* r} + a_2^* A_2^* \frac{e^{-K_2^* r}}{K_2^* r} \end{aligned} \tag{6}$$



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To evaluate A_1 , A_2 , A_1^* , A_2^* , we use the conservation equations,

which for two groups become:

$$\begin{aligned} \sigma_1 f_{11} \int_0^a N_1 r^2 dr + \sigma_2 f_{12} \int_0^a N_2 r^2 dr + \sigma_1^* f_{11}^* \int_a^\infty N_1^* r^2 dr + \sigma_2^* f_{12}^* \int_0^\infty N_2^* r^2 dr &= 0 \\ \sigma_2 f_{22} \int_0^a N_2 r^2 dr + \sigma_1 f_{21} \int_0^a N_1 r^2 dr + \sigma_2^* f_{22}^* \int_a^\infty N_2^* r^2 dr + \sigma_1^* f_{21}^* \int_0^\infty N_1^* r^2 dr &= 0 \end{aligned} \quad (7)$$

The other two equations needed are obtained by substituting (4) and (6) into (1).

Our problem is now fully determined.

In solving (3) for k_2 of (4) it often happens that no such value exists. This difficulty has never been fully understood. The most successful method to date is the so-called "hill" method. Equations (1) and (7) are transformed into equations appropriate to a subcritical system by introducing a parameter λ which becomes 1 for the critical system (see section IV for further details). For $\lambda^{-1} = 0$, both k_1 and k_2 exist. As λ^{-1} increases k_1 and k_2 change continuously until a maximum λ_m^{-1} is reached. A further increase in λ^{-1} will give a value of k_1 but no value of k_2 . The "hill" solution is then to use k_2 and a_2 corresponding to λ_m^{-1} and k_1 and a_1 for $\lambda = 1$.

As a test of our method we will give a numerical example and compare it to other reliable methods. The case chosen is a critical 25 core with an infinite WC tamper. This problem had been already solved by the spherical-harmonic method⁴⁾. The critical radius as obtained by the asymptotic method described above is $\sigma_1 a = 1.2757$. By the spherical-harmonic method, the value of $\sigma_1 a$ is 1.2841 for the P_3 approximation. Lastly we give the value obtained by the Feynman method: $\sigma_1 a = 1.2767$. We see that these three estimates of the critical radius are in good agreement. In Fig 1, we have a plot of neutron flux nv in core and tamper

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for the two velocity groups. A comparison there is made of the spherical-harmonic solution to our asymptotic method. Away from the core-tamper interface, the two methods agree fairly well. At the interface the asymptotic method gives its typical discontinuous jump in neutron flux. If a smoothing process is used on the asymptotic solution, quite good neutron distribution curves are obtained, i.e. the agreement between the asymptotic and spherical-harmonic methods becomes surprisingly good.

II Multiplication Theory for Untamped Spheres.

Suppose that we have an untamped sphere of active material of critical size. If n neutrons are introduced into the sphere distributed in the normal mode, then these neutrons will produce enough neutrons to compensate for those that leak out, and the n neutrons will be maintained in the normal mode. In this stationary process the n neutrons produce an infinite number of neutrons which leak out (neglecting depletion of the material).

We define the multiplication, M , as $1 +$ the total number of neutrons produced by each neutron. For the critical case the multiplication is infinite.

If the sphere is sub-critical, and n neutrons are introduced, they will not maintain themselves, but will leak away. The total number produced is not sufficient to compensate for the leakage. The leakage will continue until no neutrons are left in the sphere. The process takes a finite time and a finite number of neutrons have been produced. The multiplication is finite.

The reciprocal of the multiplication, $1/M$, is always finite, being 1 for a bare source and zero for the critical size. It has been found that the graph of $1/M$ vs. the radius of the active material is for 25 metal and 49 metal very closely a straight line. Because of this it is possible by measuring the

3. In the case of 49 the curve becomes slightly concave near the critical point.

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multiplication of a few sub-critical sizes to extrapolate accurately to the critical size.

The experiment is usually carried out by putting a steady source at the center of the sphere and measuring with a flat counter the total number of neutrons that emerge. The ratio of the total number that emerge to the total number introduced by the source in the same time interval is the multiplication of a central source.

We will now show how this quantity may be calculated in the approximation that the spectrum of neutrons can be taken in the form of two monochromatic groups.

Let the position vector of any point relative to the center of the sphere be denoted R , and let $|R| = r$. Let $C(R)$ be the number of collisions per sec per unit volume at the point R . In order to carry out our two-group calculation, we represent $C(R)$ by a one-column matrix with the elements $C_1(R)$ and $C_2(R)$. $C_1(R)$ is to represent the density of collisions per sec. by fast neutrons, and $C_2(R)$ the same quantity for slow neutrons. The transpose of C , C^+ , will be a one-row matrix with the elements C_1 and C_2 . In symbols we have

$$C(R) = \begin{pmatrix} C_1(R) \\ C_2(R) \end{pmatrix} \quad C^+(R) = \begin{pmatrix} C_1(R) & C_2(R) \end{pmatrix}$$

The sum of the elements of any row or column matrix, M , will be denoted $\sum M$.

Consider for the present the general type of source distribution, $S(R)$. $S(R)$ is again a one-column matrix, and its two components give the number of fast and slow neutrons produced by the source per unit volume at R per sec. The order of the components is the same as in $C(R)$.

We can now write down the matrix equation which describes the production

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and diffusion of neutrons in our system.

$$C(R) = \int K(R, R') \{FC(R') + S(R')\} dR' \quad (8)$$

This inhomogeneous integral equation expresses essentially the conservation of neutrons. F and K are square matrices of the second order $FC(R')$ is the source of neutrons due to the collisions taking place at R' . $K(R, R')$ is known as the kernel and represents the density of collisions at R due to a neutron produced at R' . The integration is over the active material and dR' is the element of volume. It is now clear that this equation states that the collisions at R are due to the neutrons produced by the source and to those produced by collisions throughout the sphere which travel to R and make a collision there. The explicit expressions for F and K are

$$F = \begin{pmatrix} 1+f_{11} & f_{12} \\ f_{21} & 1+f_{22} \end{pmatrix} \quad K(R, R') = \begin{pmatrix} \frac{\sigma_1 e^{-\sigma_1 |R-R'|}}{4\pi |R-R'|^2} & 0 \\ 0 & \frac{\sigma_2 e^{-\sigma_2 |R-R'|}}{4\pi |R-R'|^2} \end{pmatrix}$$

We see that K is a diagonal matrix. It follows that $K^t = K$. We also have the relation $K(R, R') = K(R', R)$ i.e. K is symmetric in R and R' . We also note that in general $F^t \neq F$.

We will show presently that the solution of eq. (8) reduces to finding two sets of characteristic functions and values defined by two homogeneous integral equations obtained from (8). The first set of characteristic functions and values is defined by the equation

$$U_i(R) = \lambda_i \int K(R, R') F U_i(R') dR' \quad i = 1, 2, 3, \dots \quad (9)$$

We call the U_i 's the direct functions. They do not form an orthogonal set as

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in the case of the one-group theory⁶⁾. We obtain what is called a bi-orthogonal set with another infinite set of functions which we call the adjoint functions. The adjoint functions are defined by the equation⁷⁾.

$$V_i(R) = \lambda_i \int K(R, R^0) F^+ V_i(R^0) dR^0 \quad i = 1, 2, 3, \dots \quad (10)$$

Equations (9) and (10) possess the same set of eigenvalues since the Fredholm determinants are the same for the two equations. It is now possible to show quite easily that the U's and V's do form a bi-orthogonal set, i.e. that we have the relation

$$\int V_j^+(R) F U_i(R) dR = \delta_{ji} \quad (11)$$

To do this we first take the transpose of eq. (10) for the j^{th} eigenfunction.

This gives

$$V_j^+(R) = \lambda_j \int V_j^+(R^0) F K(R, R^0) dR^0 \quad (12)$$

Now we multiply eq. (9) by $\lambda_j V_j^+(R) F$ from the left and subtract from this eq. (12) multiplied by $\lambda_i F U_i(R)$ from the right. Then we integrate over the active material.

We have

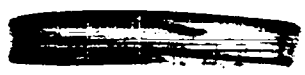
$$(\lambda_j = \lambda_i) \int V_j^+(R) F U_i(R) dR = \lambda_j \lambda_i \left\{ \iint dR dR^0 V_j^+(R) F K(R, R^0) F U_i(R^0) - \iint dR dR^0 V_j^+(R^0) F K(R, R^0) F U_i(R) \right\}$$

Since K is symmetric, and the names of the variables do not matter, the right side vanishes. Assuming no degeneracy, that is for $i \neq j$ $\lambda_i \neq \lambda_j$, we find that

6). Cf. LA. 465

7). In the general case, the definition of the adjoint is $V_i(R) = \lambda_i \int K^+(R^0, R) F^+ V_i(R^0) dR^0$. This reduces to (10) in our case.

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$\int V_j^+(R) F U_i(R) dR = 0$. For $i = j$ we may normalize the integral to unity⁸⁾.

By means of the bi-orthogonal set we can Fourier analyze any density function in terms of normal modes. However, the usefulness of the set goes further than that. For it is also possible to analyze any square matrix (of second order) in terms of the direct and adjoint functions. In particular we easily find that

$$K(R, R^0) = \sum_i \frac{U_i(R) V_i^+(R^0)}{\lambda_i} \quad (13)$$

This is easily checked by substituting the series for K in eq. (2). The series for K is called the bilinear series for the kernel. It will be found very useful.

We now go back to the solution of eq. (8). The experiment which we are considering has a central source which let us say emits q_1 fast neutrons and q_2 slow neutrons per second where $q_1 + q_2 = 1$. It is useful to define the matrix

$Q^+ = \begin{pmatrix} q_1 & \\ & q_2 \end{pmatrix}$. We have then that

$$S(R) = \begin{pmatrix} q_1 \frac{\delta(r)}{4\pi r^2} \\ q_2 \frac{\delta(r)}{4\pi r^2} \end{pmatrix}$$

We can now perform the integration of the kernel over the source in eq. (8). We obtain two expressions, one by using the matrix for K and the other by using the bilinear series for K. We have

$$S^0(R) \equiv \int K(R, R^0) S(R^0) dR^0 = \begin{pmatrix} q_1 \sigma_1 \frac{e^{-\sigma_1 r}}{4\pi r^2} \\ q_2 \sigma_2 \frac{e^{-\sigma_2 r}}{4\pi r^2} \end{pmatrix} = \sum_i \frac{U_i(R) (V_i^+(0) Q)}{\lambda_i} \quad (14)$$

We note that $(V_i^+(0) Q)$ is a scalar quantity. To solve eq. (8) expand $C(R)$ as an infinite series in the U_i 's.

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8) In this way there remains an arbitrariness in the way the U's and V's are normalized separately. This is of no importance as in the applications it is always the product of the two that appears.

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$$C(R) = \sum_i C_i U_i(R)$$

Substituting in eq. (8) and using eq. (6) we have

$$\sum_i C_i U_i = \sum_i \frac{(V_i^+ (0) Q) U_i}{\lambda_i} + \sum_i \frac{C_i U_i}{\lambda_i}$$

This equation can be solved for the C's by multiplying by $V^+ F$ from the left and integrating over the active material. We find that

$$C_i = \frac{V_i^+ (0) Q}{\lambda_i - 1} = \frac{V_i^+ (0) Q}{\lambda_i} + \frac{V_i^+ (0) Q}{\lambda_i (\lambda_i - 1)} \quad (15)$$

Using eq. (14), we have as the final solution of eq. (8)

$$C = \left(\begin{array}{c} q_1 \sigma_1 \frac{e^{-\sigma_1 r}}{4\pi r^2} \\ q_2 \sigma_2 \frac{e^{-\sigma_2 r}}{4\pi r^2} \end{array} \right) + \sum_i \frac{(V_i^+ (0) Q) U_i(R)}{\lambda_i (\lambda_i - 1)}$$

In this form of the solution, the first collision which is singular has been separated from the infinite sum. In this way the convergence of the series is made more rapid and fewer characteristic functions and values have to be calculated.

The total number of collisions which take place in a sphere of radius

a is

$$\int_0^a C dR = \left(\begin{array}{c} q_1 (1 - e^{-\sigma_1 a}) \\ q_2 (1 - e^{-\sigma_2 a}) \end{array} \right) + \sum_i \frac{(V_i^+ (0) Q) \int_0^a U_i(R) dR}{\lambda_i (\lambda_i - 1)} \quad (16)$$

The total number of fissions taking place in the sphere is $(\sigma_f / \sigma_1) (\sigma_{f2} / \sigma_2) \int_0^a C dR$, and the multiplication is $1 + \mathcal{I}(F-1) \int_0^a C dR$. Here \mathcal{I} stands for the identity or unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

III. Multiplication Theory of Tamped Spheres.

One of the most important measurements made by the project was the determination of the critical sizes of tamped spheres of 25 and 49. The principal

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tampers used were tungsten carbide and tuballoy. The critical size was approached by measuring the multiplication of sub-critical spheres. It was predicted theoretically and checked experimentally that the graph of $1/M$ vs. the radius of the sphere is a straight line from reasonable large radii to the critical radius.

In order to measure the multiplication of tamped spheres, it is necessary to place small foils in the active material itself. In this way one usually obtains the total number of fissions taking place in the sphere. This can then be converted into the multiplication by comparing the total number of fissions and the multiplication in the case of untamped spheres⁹⁾.

The calculation of the multiplication of tamped sub-critical spheres is very similar to the calculation for the untamped spheres, particularly in some of the more formal aspects. There are nevertheless some instances where the theory requires important elaboration.

We consider a spherical core of radius a , with a unit source at the center surrounded by an infinite tamper. Let us introduce the matrix G which plays the same role in the tamper as F does in the core. The inhomogeneous integral equation which describes the production and diffusion of neutrons in our assembly now becomes

$$C(R) = \int_c K(R, R') \left\{ FC(R') + S(R') \right\} dR' + \int_t K(R, R') GC(R') dR' \quad (17)$$

In this equation $C(R)$ is the density of collisions per sec. at any point in the core or tamper. The letters c and t under the integral signs indicate that the integrations are to be taken over the core and tamper respectively. G is a square matrix of second order as follows:

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$$G = \begin{pmatrix} 1+f_{11}^* & f_{12}^* \\ f_{21}^* & 1+f_{22}^* \end{pmatrix}$$

The kernel $K(R, R')$ = $\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ where

$$K_i(R, R') = \sigma_i(R) \frac{\int_0^{|R-R'|} \sigma_i dx}{4\pi |R-R'|^2} \quad i = 1, 2 \quad (18)$$

The integration in the exponent is to be taken along the straight line between R and R' . This complexity of the kernel is necessitated by the fact that the core and tamper are in general of different transport mean free path.

Let us integrate eq. (17) over all space. Using the fact that

$$\int_{C+T} K(R, R') dR = 1, \text{ we have}$$

$$(F-I) \int_C C(R) dR + \int_C S(R) dR = -(G-I) \int_T C(R) dR \quad (19)$$

This is the statement of the conservation of neutrons in two groups.

It is possible to write eq. (17) in a form in which the tamper never appears explicitly as suggested by R. Feynman. For that purpose we introduce the kernel $\mathcal{K}(R, R')$ which is defined as follows:

$$\mathcal{K}(R, R') = K(R, R') + \int_T K(R, R^1) GK(R^1, R') dR^1 + \iint_{TT} K(R, R^1) GK(R^1, R^2) GK(R^2, R') dR^1 dR^2 + \dots \quad (20)$$

In this equation R and R' are restricted to the core. What does this kernel represent? It represents the density of collisions at a point R in the core due to a neutron released at a point R' in the core. The successive terms in the infinite series are just the density of collisions at R when a neutron goes from R' to R making respectively 0, 1, 2 ... collisions in the tamper. In a more formal fashion one is led to define such a kernel by applying the method of successive

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iteration to eq. (17) in the following way. As a first approximation for $C(R)$ neglect the second term, i.e. the integral over the tamper. Then substitute this for $C(R)$ in the tamper integral. This gives the second approximation for $C(R)$ as follows

$$C(R) = \int_C \left\{ K(R, R^0) + \int_T K(R, R^1) GK(R^1, R^0) dR^1 \right\} \{ FC(R^0) + S(R^0) \} dR^0$$

If this process is continued one is led to the integral equation

$$C(R) = \int_C \mathcal{K}(R, R^0) \{ FC(R^0) + S(R^0) \} dR^0 \quad (21)$$

where $\mathcal{K}(R, R^0)$ is given by eq. (20). In this form, the equation for the tamped assembly is the same as for the untamped assembly, eq. (8). Indeed all the theory of solving the inhomogeneous equation already developed can now be taken over to the tamped case. Let us just look a little into the new kernel. We consider it as a sum of kernels $\mathcal{K}_n(R, R^0)$ i.e. $\mathcal{K}(R, R^0) = \sum_n \mathcal{K}_n(R, R^0)$ where $\mathcal{K}_1(R, R^0) = K(R, R^0)$ and $\mathcal{K}_n(R, R^0) = \int_T \dots \int_T K(R, R^1) GK(R^1, R^2) G \dots K(R^{n-2}, R^{n-1}) GK(R^{n-1}, R^0) dR^1 \dots dR^{n-1}$.

\mathcal{K}_n is called the nth iterated kernel. \mathcal{K} is not symmetric in R and R^0 , nor is \mathcal{K} a symmetric matrix. These properties are lost in going from the untamped to the tamped case. A useful recursion relationship for the iterated kernels is the following

$$\mathcal{K}_n(R, R^0) = \int_T \mathcal{K}_{n-1}(R, R^1) GK(R^1, R^0) dR^1 \quad (22)$$

A more general recursion relationship is the following

$$\mathcal{K}_n(R, R^0) = \int_C \mathcal{K}_2(R, R^1) G \mathcal{K}_2(R^1, R^0) dR^1$$

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We have seen that in the untamped case, the inhomogeneous equation could be solved once we had found the bi-orthogonal set of the homogeneous equation. The equations we had to solve were eqs. (9) and (10). The kernels in these equations are essentially Milne kernels which we know how to treat (see below IV). Hence the problem is essentially solved.

In the tamped case the homogeneous equation for the direct functions and the characteristic values is

$$U_i(R) = \lambda_i \int_c \mathcal{K}(R, R') FU_i(R') dR' \quad i = 1, 2, \dots \quad (23)$$

This equation now takes the place of eq. (9). Here \mathcal{K} is no longer a Milne-type kernel and we must now say how the characteristic functions and values of this kernel are to be found.

We now assert that the characteristic functions and values of eq. (23) can be found by solving the equation

$$U_i(R) = \lambda_i \int_c K(R, R') FU_i(R') dR' + \int_c K(R, R') GU_i(R') dR' \quad i = 1, 2, \dots \quad (24)$$

To prove that this is so, first substitute the series for $\mathcal{K}(R, R')$. We have

$$U_i(R) = \sum_{n=1}^{\infty} \lambda_i \int_c \mathcal{K}_n(R, R') U_i(R') dR'$$

Now substitute the solution of eq. (24) into this equation. Using the recursion relation eq. (22) we have

$$\lambda_i \int_c \mathcal{K}_n(R, R') FU_i(R') dR' = \int_c \mathcal{K}_{n-1}(R, R') GU_i(R') dR' - \int_c \mathcal{K}_n(R, R') GU_i(R') dR' \quad n \geq 2$$

$$U_i(R) = \int_c \mathcal{K}_2(R, R') GU_i(R') dR' \quad n=1$$

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If we now sum over all n , we see that all the terms cancel except $U_i(R)$.

If eq. (24) be integrated over all space, we obtain the conservation law for the characteristic functions

$$(\lambda_i F - I) \int_C U_i(R) dR = -(G - I) \int_C U_i(R) dR \quad (25)$$

The adjoint system of characteristic functions is defined by the integral equation

$$V_j(R) = \lambda_j \int_C K^+(R', R) F^+ V_j(R') dR' \quad j = 1, 2, \dots \quad (26)$$

In this way it follows in a manner completely analogous to the untamped case that

$$\int_C V_j^+(R) F U_i(R) dR = \delta_{ji}$$

This is true of course provided the direct and adjoint set of functions possess the same set of characteristic values as is written into eq. (26). That the two sets of characteristic values are the same can be seen in the following way. Let us write eq. (24) in the form

$$U_i(R) = \int_{C+t} K(R, R') L(R') U_i(R') dR'$$

where $L(R')$ equals $\lambda_i F$ in the core and G in the tamper. Consider now an adjoint equation defined in the following way. $U_i(R) = \int_{C+t} L^+(R) K(R', R) U_i(R') dR'$. The direct and adjoint equation possess the same set of characteristic values since they have the same Fredholm determinant. The adjoint equation can now be transformed into our adjoint equation by multiplying by $K(R'', R)$ and integrating over all R . This proof and the proof in the case of the untamped sphere are not completely vigorous since our kernels are actually logarithmically singular for $r = r'$. This

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can be remedied by always considering the average of the kernel over a small region in setting up the determinants. In this way all the quantities are finite.

It is now necessary to look into the nature of the kernel $\mathcal{K}^+(R^i, R)$.

We have that

$$\mathcal{K}^+(R^i, R) = \sum_n \mathcal{K}_n^+(R^i, R) \text{ and } \mathcal{K}_n^+(R^i, R) = \int_{\mathcal{C}} \dots \int_{\mathcal{C}} K(R^{n-1}, R) G^+ K(R^{n-2}, R^{n-1}) \dots G^+ K(R^i, R^1) dR^1 \dots dR^{n-1}$$

We can now see that the equation we must solve for the adjoint characteristic function is

$$V_j(R) = \lambda \int_{\mathcal{C}} K(R^i, R) F^+ V_j(R^i) dR^i + \int_{\mathcal{C}} K(R^i, R) G^+ V_j(R^i) dR^i \quad j=1, 2 \dots \quad (27)$$

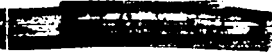
For if we now employ the method of iteration to remove the tamper term, we are led to equation (26).

It is of course possible to prove directly that the U's and V's form a bi-orthogonal set from eqs. (24) and (27) which contain the tamper terms explicitly. To do this multiply eq. (24) by $\lambda_j V_j^+(R) F$ from the left and then subtract from this the transpose of eq. (27) multiplied by $\lambda_i F U_i(R)$ from the right. Then integrate both sides of the resulting equation over the core. We have

$$(\lambda_j - \lambda_i) \int_{\mathcal{C}} V_j^+ F U_i dR = \lambda_j \lambda_i \left\{ \iint_{\mathcal{C}\mathcal{C}} dR dR^i V_j^+(R) F K(R, R^i) F U_i(R^i) - \iint_{\mathcal{C}\mathcal{C}} dR dR^i V_j^+(R^i) F K(R^i, R) F U_i(R) \right\} + \lambda_j \int_{\mathcal{C}} dR \int_{\mathcal{C}} dR^i V_j^+(R) F K(R, R^i) G U_i(R^i) - \lambda_i \int_{\mathcal{C}} dR \int_{\mathcal{C}} dR^i V_j^+(R^i) G K(R^i, R) F U_i(R)$$

The first two terms on the right cancel. To show that the last two terms on the right cancel we rewrite them changing the orders of integration in each term and substituting from eqs. (24) and (27). We have for these terms

$$\int_{\mathcal{C}} dR^i \left\{ V_j^+(R^i) - \int_{\mathcal{C}} V_j^+(R^i) G K(R^i, R^i) dR^i \right\} G U_i(R^i) - \int_{\mathcal{C}} dR^i V_j^+(R^i) G \left\{ U_i(R^i) - \int_{\mathcal{C}} K(R^i, R^i) G U_i(R^i) dR^i \right\}$$



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It is now clear that these terms also cancel. We have then that if $\lambda_j \neq \lambda_i$ that $\int_C V_j^* F U_i dR = 0$. For $i = j$, $\lambda_i = \lambda_j$ this integral can be normalized to unity.

Let us define the matrix $\sigma(R) = \begin{pmatrix} \sigma_1(R) & 0 \\ 0 & \sigma_2(R) \end{pmatrix}$ and let us denote by σ_c and σ_t this matrix when it is evaluated in the core and tamper respectively. From eq. (18) and the definition of $K(R', R)$ we see that we have the following matrix relation $\sigma(R)K(R', R) = \sigma(R')K(R, R')$. Let us now multiply eq. (27) by $\sigma(R)$ and then integrate over all space. We obtain the conservation law for the adjoint functions.

$$\sigma_c (\lambda_j F^+ - I) \int_C V_j(R) dR = - \sigma_t (G^+ - I) \int_t V_j(R) dR \quad (28)$$

We see that this relation has a slightly different structure from the conservation law for the direct functions, eq. (25). Specifically there is an "extra" σ_c on the left and an "extra" σ_t on the right side of the equation. Let us look into this a little. We go back to eq. (24), the integral equation for the direct functions. This equation has direct physical meaning as follows: $U_i(R') dR'$ is the number of collisions in dR' . In each collision $\lambda_i F$ neutrons come off in the core and G in the tamper. A neutron so produced makes $K(R, R') dR$ collisions in dR . And the integral equation is the condition that the system is then maintained. It is then immediately clear without the process of integrating over all space that the conservation law is just eq. (25). Let us now turn to the adjoint functions defined by eq. (27). As the equation stands, it has no direct physical meaning. The reason is that $K(R', R)$ says that the collision is made at R' and not at R . We will now write eq. (27) so that one can interpret it. Multiply eq. (27) by $\sigma(R)$ and use the relation that $\sigma(R)K(R', R) = K(R, R')\sigma(R')$. Now introduce between F^+ and $V(R')$ and between G^+ and $V(R')$ the unit matrix $\sigma^{-1}(R')\sigma(R')$. Here $\sigma^{-1}(R')$

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denotes the reciprocal of $\sigma(R')$. This is permissible since $|\sigma(R')| \neq 0$. We obtain the following equation

$$\sigma(R) V_j(R') = \lambda_j \int_c K(R, R') \sigma(R') F^+ \sigma^{-1}(R') \left[\sigma(R') V_j(R') \right] dR' + \int_t K(R, R') \sigma(R') G^+ \sigma^{-1}(R') \left[\sigma(R') V_j(R') \right] dR'$$

This equation can be interpreted in the same way as eq. (24). We see however that $\sigma(R') V_j(R')$ plays the role of $U_i(R')$, $\lambda_j \sigma(R') F^+ \sigma^{-1}(R')$ the role of $\lambda_i F$ and $\sigma(R') G^+ \sigma^{-1}(R')$ the role of G . With this interpretation we write down the conservation law immediately

$$(\lambda_j \sigma_c F^+ \sigma_c^{-1} - I) \int_c \sigma_c V_j(R) dR = -(\sigma_t G^+ \sigma_t^{-1} - I) \int_t \sigma_t V_j(R) dR$$

This is exactly eq. (28) if we take σ_c and σ_t from under the integral signs as we may and put these matrices in front of the respective brackets. The physical interpretation of the adjoint equation is necessary when we depend on solving an integral equation by intuitive rather than formal methods.

The bilinear series for the kernel $\mathcal{K}(R, R')$ has the same form as in an untamped case.

$$\mathcal{K}(R, R') = \sum_{i=1}^{\infty} \frac{U_i(R) V_i^+(R')}{\lambda_i}$$

and the solution of the inhomogeneous integral equations (17) and (21) is given by eq. (15)

$$C = \sum_t \left(\frac{V_i^+(0) \gamma}{\lambda_i} \right) U_i(R) + \sum_t \left(\frac{V_i^+(0) Q}{\lambda_i (\lambda_i - 1)} \right) U_i(R)$$

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The first sum on the right is due to neutrons making their first collision in the core. The second term contains all the higher collisions. The first term contains the singular piece

$$S^0(R) = \begin{pmatrix} q_1 \sigma_1 \frac{e^{-\sigma_1 R}}{4\pi R^2} \\ q_2 \sigma_2 \frac{e^{-\sigma_2 R}}{4\pi R^2} \end{pmatrix}$$

This is the contribution of true first collisions as distinct from first collisions in the core. It is again useful to separate this out. We have then for the final solution of the inhomogeneous integral equation

$$C(R) = S^0(R) + \sum_i \left\{ \frac{v_i^+(0)Q}{\lambda_i(\lambda_i-1)} + \left(\frac{v_i^+(0)Q}{\lambda_i} - \int_0^R v_i^+(R')FS^0(R')dR' \right) \right\} U_i(R) \quad (29)$$

The multiplication is $1 + \mathcal{J}(F-I) \int_0^R C dr$

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IV. Characteristic Functions and Values

We still have to determine the U_i , V_i and λ_i of (16) and (29). These characteristic functions and values are determined by the integral equations (9) and (12) for the untamped case; (24), (25), (27) and (28) for the tamped case. If we remember that $C_1 \equiv N_1 \sigma_1$ and $C_2 \equiv N_2 \sigma_2$ and that $G = 0$ for an untamped core, then (9) and (24) become, if evaluated for the central point ($R = 0$), the same as (1) except that the F matrix of the core is multiplied by the characteristic value λ_i . A similar relation exists between the conservation equations (25) or (28) and (7). Thus we observe that the transport and conservation equations for any characteristic mode differs from the corresponding critical equations in that the F quantities of the core are multiplied by λ_i . The problem of the critical system is: given for the lowest and only mode that $\lambda_i = 1$, we seek the value of the core radius "a" to satisfy our fundamental equations. On the other hand, the subcritical problem is: given the core radius "a", we wish to determine $\lambda_1, \lambda_2, \dots$. We shall order the characteristic values by taking $\lambda_1 \leq \lambda_2 \leq \dots$. For a subcritical system $\lambda_1 > 1$. We give a physical argument for this assertion. When the system is subcritical, the values of $1 + f_{11}, f_{12}, \dots$ entering F are not sufficiently large to maintain the system critical. But the subcritical equations for a mode are essentially critical equations with $1 + f_{11}, f_{12}, \dots$ changed into $\lambda_1(1 + f_{11}), \lambda_1 f_{12}, \dots$. Thus $\lambda_1 > 1$.

For an assumed value of λ_i , the asymptotic equations can be written as in (3). (Of course F has now become $\lambda_i F$). For a given value of λ_i , we then get two values of k . It is more convenient to plot $1/\lambda$ against k^2 . When k^2 becomes negative, i.e., k is pure imaginary, the arctans of (3) are changed into arctanhs as has been pointed out in section I. We sketch in Fig. 2 a typical case of these curves. Curve A, the upper branch, starts from $k^2 = \infty, 1/\lambda = 0$

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and rises monotonically to $1/\lambda = \infty$ for $k^2 = -\sigma_1^2$. Curve B likewise ascends smoothly. The curves are not extended to the left of $k^2 = -\sigma_1^2$ as no consistent theory has yet been evolved for $k^2 < -\sigma_1^2$. (We have assumed $\sigma_1 \leq \sigma_2$)

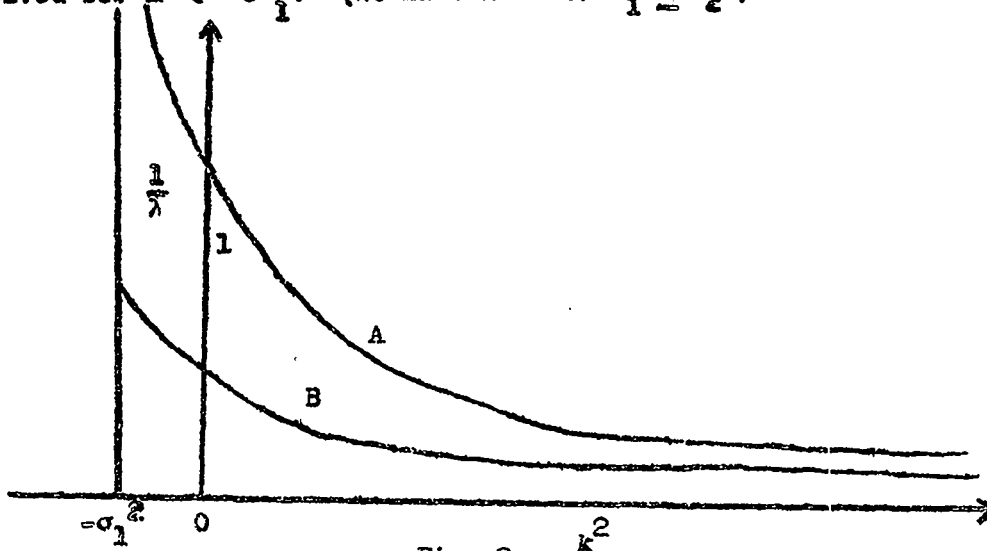


Fig. 2

With the aid of Fig. 2, we can discuss the general behavior of the characteristic modes for any subcritical system. Let us start with an extremely small core radius ($a \rightarrow 0$). Then $\lambda_1 \rightarrow \infty$ or $1/\lambda_1 \rightarrow 0$. In Fig. 3, we draw the lines for λ_1, λ_2 , etc. We see that values of k^2 for λ_1, λ_2 , etc. are all positive. Hence for this case, the characteristic functions of each mode will have solutions of the form (4) with the $\sinh(k_2 r)$ there changed to a $\sin(k_2 r)$.

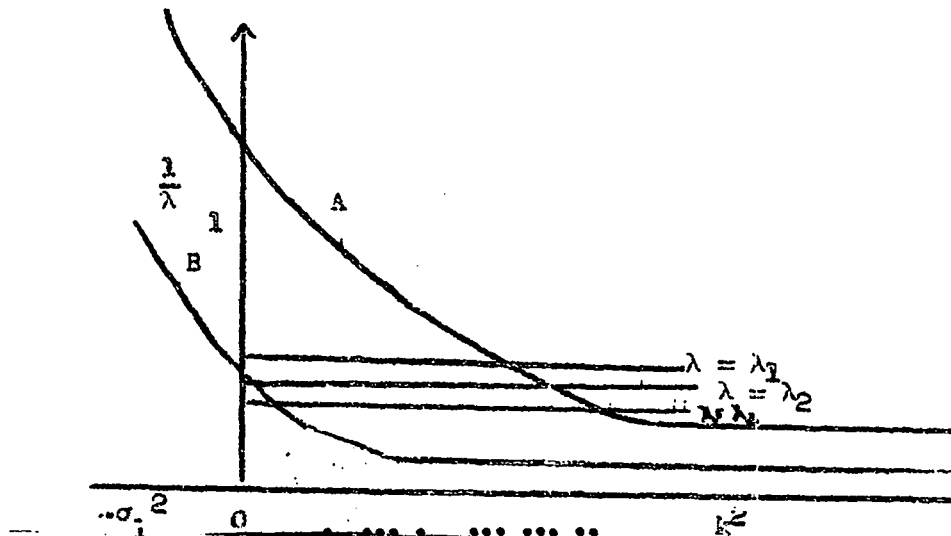


Fig. 3

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As the radius of the core is increased, λ_1 decreases. As an intermediate case let us take the situation indicated in Fig. 4. For the lowest mode (λ_1) the values of k^2 are positive and negative. For this mode the characteristic functions have the form (4). The higher modes λ_2, λ_3 have the values of k^2 both positive and therefore their characteristic functions will be linear combinations of $(\sin k_1 r)/(k_1 r)$ and $(\sin k_2 r)/(k_2 r)$.

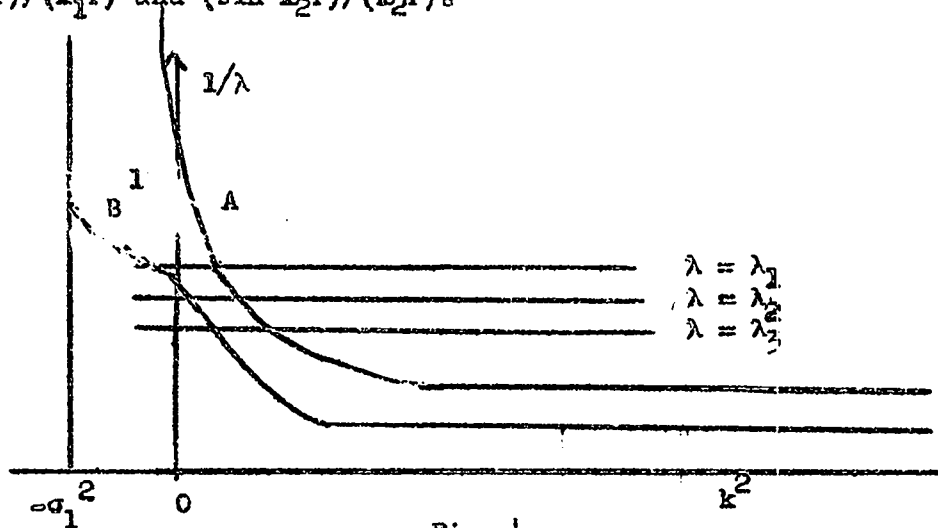


Fig. 4

We discuss a further example (Fig. 5). The core has become still larger. The value λ_1 has now reached is such that no value of k^2 exists on the B curve. This is certainly an anomalous situation. No satisfactory solution is yet known. We have in the meantime adopted the "hill" approximation, i.e. that $k_2^2 = -\sigma_1^2$, or that we use the nearest value of $1/\lambda$ which gives a solution, namely the point H, the highest point on the curve B. The value of k_1^2 always exists and is here positive. This type of solution was discussed at the end of section I. The higher modes behave properly. The second mode is here a mixed sin and sinh. The third mode is a sin and sin combination etc.

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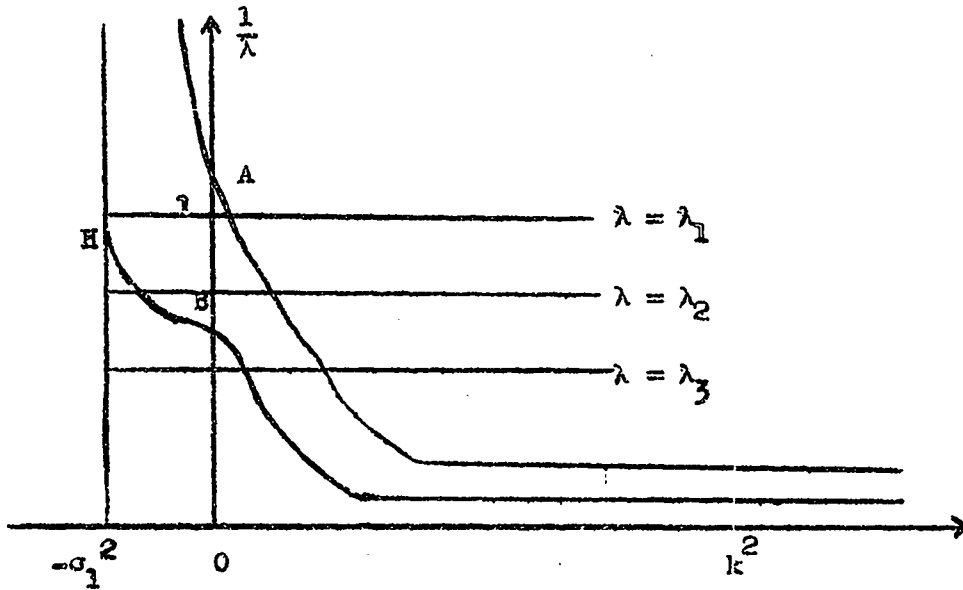
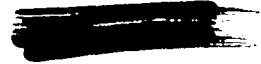
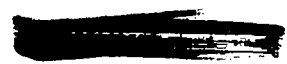


Fig. 5

The final example is that of the critical case. The lowest or fundamental mode is usually of the type indicated in Fig. 5. The higher modes need not be considered as their contribution to the multiplication or neutron density is negligible. The mathematical explanation is the presence of $(\lambda_1 - 1)^{-1}$ in these expressions and $\lambda_1 \rightarrow 1$ as criticality is approached. See (16) and (29).

In practice to find the lowest characteristic value λ_1 for a subcritical system, we guess a value of λ_1 (of course higher than one), and the larger, the smaller the core radius. With this value of λ_1 , we then find k_1, a_1 and k_2, a_2 from the revised (3) ($F \rightarrow \lambda_1 F$). K_1^*, a_1^* and K_2^*, a_2^* are determined from (5). The characteristic functions are then of the form (4) and (6) or variations of them as outlined above. When the latter functions are substituted in the transport and conservation equations (24) and (25) or (26) and (28), we get four equations (two for the untamped; (9) or (12)). Eliminating the arbitrary coefficients A_1, A_2, a_1^*, a_2^* , we get a determinant, Δ , which

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should vanish. By evaluating several values of Δ for the corresponding assumed values of λ_1 , we can finally interpolate to the correct value of λ_1 to any accuracy. Once the lowest mode (λ_1) is obtained, the trial λ is increased. The next higher value of λ to give a solution to our equations is called λ_2 . In this way as many modes as desired can be calculated. A few such modes usually suffice.

The determination of λ_1 , U_1 , V_1 can thus be carried out. A check of the method should be that λ_1 obtained from U_1 is the same as that from V_1 . However, a slight discrepancy may exist between the two values due to the fact that the characteristic functions employed are only approximate and not exact solutions of our equations.

V. Results.

We shall first report our results for the multiplication experiments on 3-1/2" and 4-1/2" diameter "25" spheres untamped or tamped with WC or U. A brief account of these experiments has already been given in LANS 227. Theoretical curves were calculated on both the one-group and two-group theories for 75% 25 of density 18.0 gm/cm³. When the reciprocal multiplication (1/M) is plotted against the core diameter, the curves shown in Fig. 6 are found to be remarkably straight. A correction (here small) was made for the presence of the central hole and finite size of the source. As seen from Fig. 6, the experimental points fit the theoretical curves well. The one group and two group methods give essentially the same result. This was found to be true for both 25 and 49 cores untamped or tamped. In fact the practical procedure is to determine the critical mass from a two group calculation and then adjust the one group constants to give the same critical mass. Thereafter the one group theory can be used to obtain the remainder of the multiplication

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curve. As a matter of record, the δ phase k_{99} calculations of LAMS 253 were done in this manner. Elaborate two-group α phase k_{99} calculations were already completed when it was decided to use the δ phase as the core material. Rather than repeat the computations for the δ phase k_{99} on the two-group method, one group was thought sufficiently accurate as based on our experience with the close check between the two methods applied to several different cases. A detailed discussion of 25 and k_{99} multiplication experiments is given in a recent report LA-464.

Another class of experiments was performed in which the radial neutron distribution as detected by 25, 28, and 37 small fission chambers were measured throughout the core and tamper of various subcritical assemblies. The experimental information is contained in LA-307 and LA-421 for a mock-fission source and in LA-402 for a thermal-fission source. The former work will henceforth be conveniently referred to as Manley's and the latter as Anderson's.

As the overall multiplication was excellently verified by the theory, it was decided that the multiplication which can be also calculated from the experimental neutron curves obtained with a 25 chamber should be normalized (if need be) to give the multiplication correctly. As a matter of fact, Anderson's multiplication values agreed with Williams'¹⁰⁾ very well. However, the multiplication as computed from Manley's curves seemed to be a little low (about 5%). The difference we attribute to the perturbation due to the finite size of the detector. In any case we adjusted the theoretical and experimental neutron distributions detected by 25 foils to have the same neutron source strength and further that the multiplication in the core (after the multiplication due to the direct source is subtracted off) should also agree. Thus for the 25 detectors, the neutron distributions in the core are only to be checked for shape, but not

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for absolute magnitude. As we see from Figs. 7, 9, 11, 13, the core distribution for 3-1/2" and 4-1/2" diameter cores with WC or U tampers is as good as can be hoped for. In order to remove the $1/r^2$ singularity near the source, we have plotted r^2F against r . F is the number of fissions per cc detected at the radial distance r . The titles of the figures are to be interpreted as follows: (a) the diameter size of the core (3-1/2" or 4-1/2") (b) the tamper (WC or U) (c) the detector (25, 28, or 37) and finally (d) the location (core or tamper). The theoretical curves are drawn as solid lines and the experimental points of LA-307 and LA-421 are indicated as Manley, and the corresponding points of LA-402 are put in as Anderson.

The tamper distributions with 25 detectors are, we wish to stress, no longer normalized or interfered with in any way. In the tamper, far away from the source, we revert to the usual F against r plot. The tamper distributions are complicated by the presence of iron at a certain distance out. A rough one group calculation was made to take account of the iron. The thus corrected distribution is shown as a dotted line in these figures. We noted that the neutron flux distribution (Fig. 1) has an abrupt transition at the core-tamper interface. These distributions as well as the fission cross sections entering the expression for F were made smooth in the transition zone between the core and tamper. When we think of the complications involved in the solution of these problems, we may say that the agreement between the theory and experiment is good.

We now turn to the distributions measured with the 28 detectors. Only one adjustment was made, namely, that the experiment should have the same number of fissions per cc near the source. We have in essence calibrated the weight of the

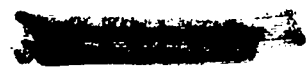
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28 foil detectors in this way. Those who know the difficulties that have been encountered with 28 foils would, we think, say that the agreement between theory and experiment is not as bad as could have been imagined.

Exactly the same procedure was followed for the 37 foils as was described for the 28 foils above.



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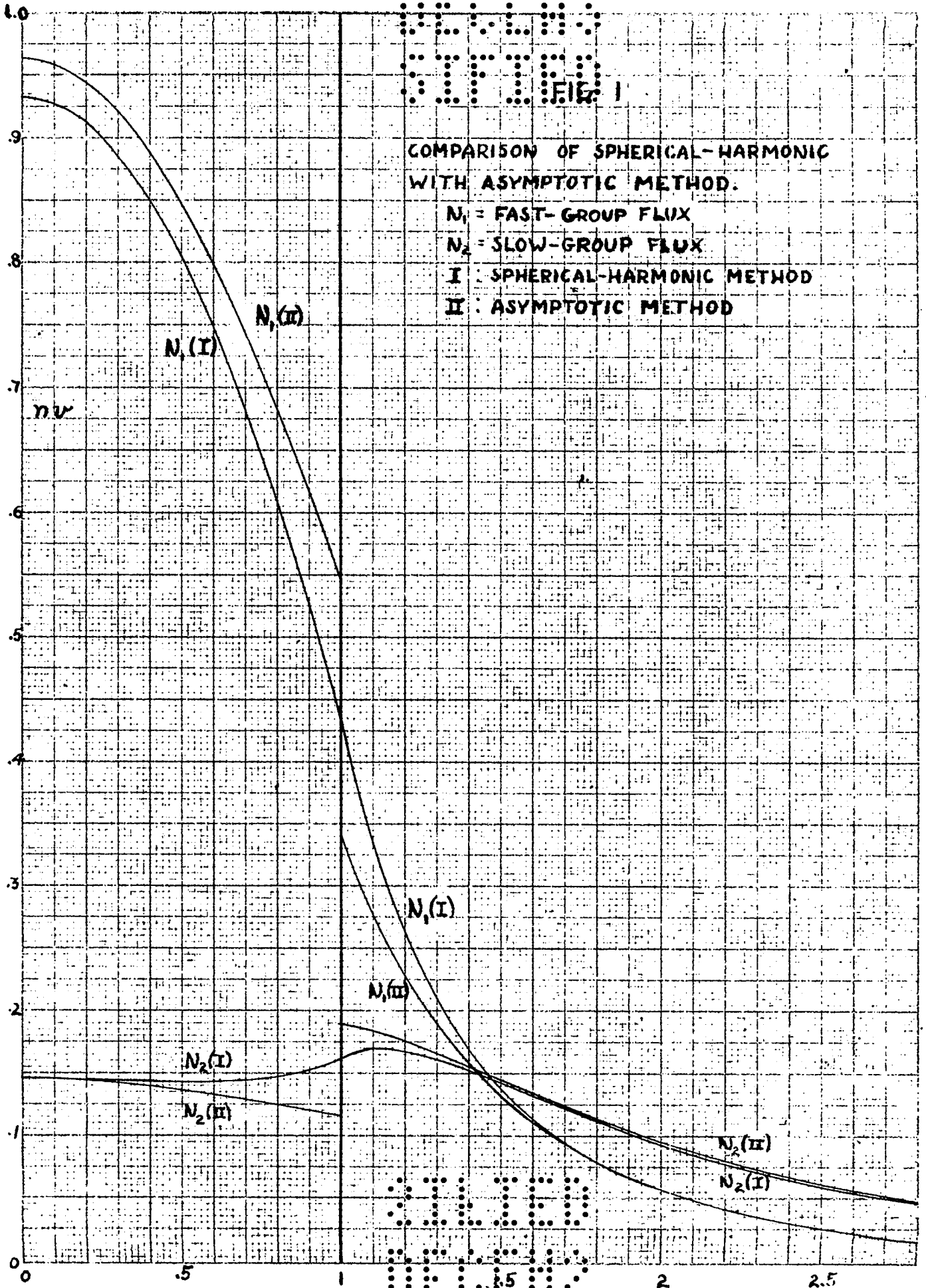
COMPARISON OF SPHERICAL-HARMONIC WITH ASYMPTOTIC METHOD.

N_1 = FAST-GROUP FLUX

N_2 = SLOW-GROUP FLUX

I : SPHERICAL-HARMONIC METHOD

II : ASYMPTOTIC METHOD

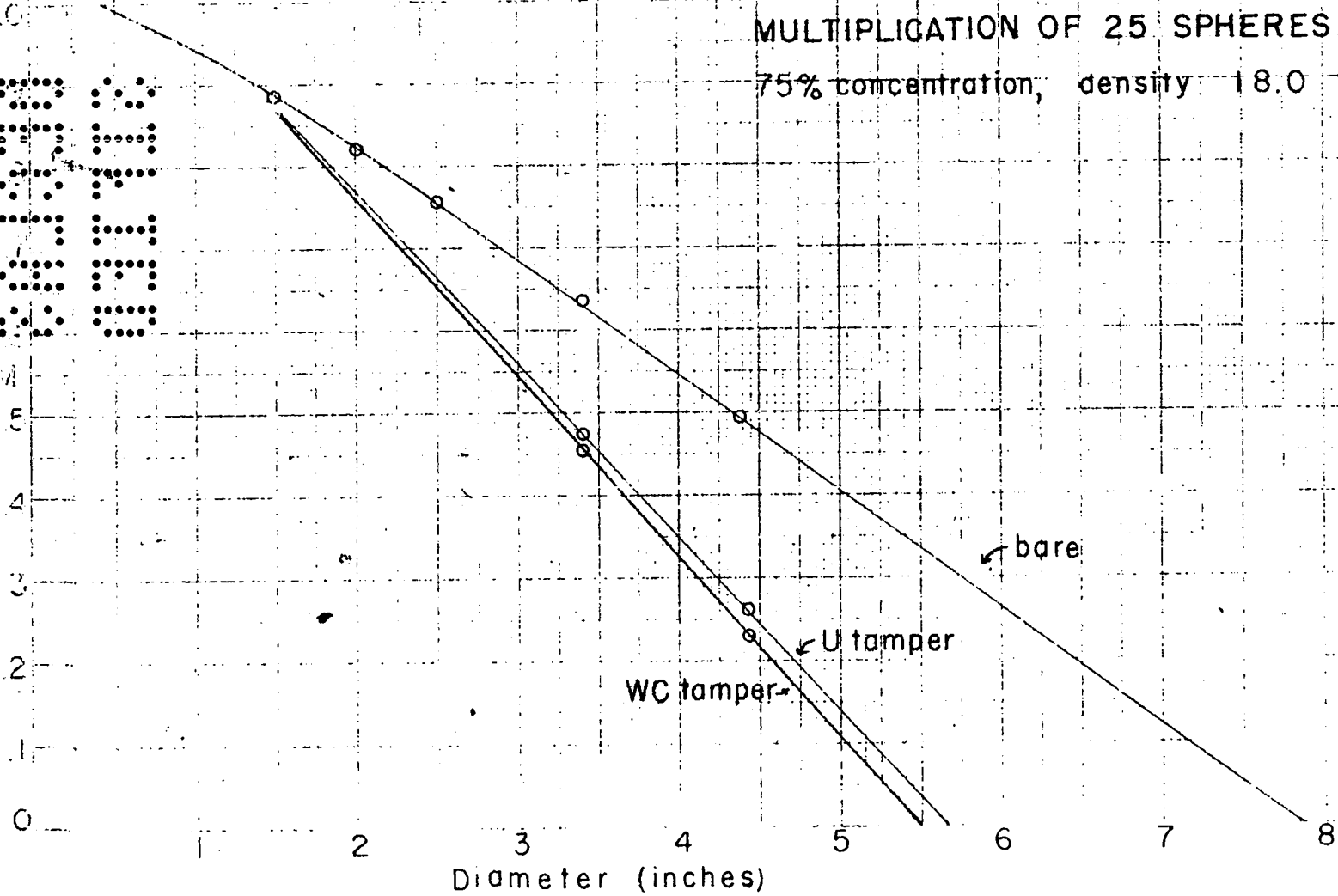


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Fig. 6

MULTIPLICATION OF 25 SPHERES
75% concentration, density 18.0



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WC tamper

U tamper

bare

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JL W/C 25
CORE
E. J. WOOD
O. HARLEY
A. ANDERSON

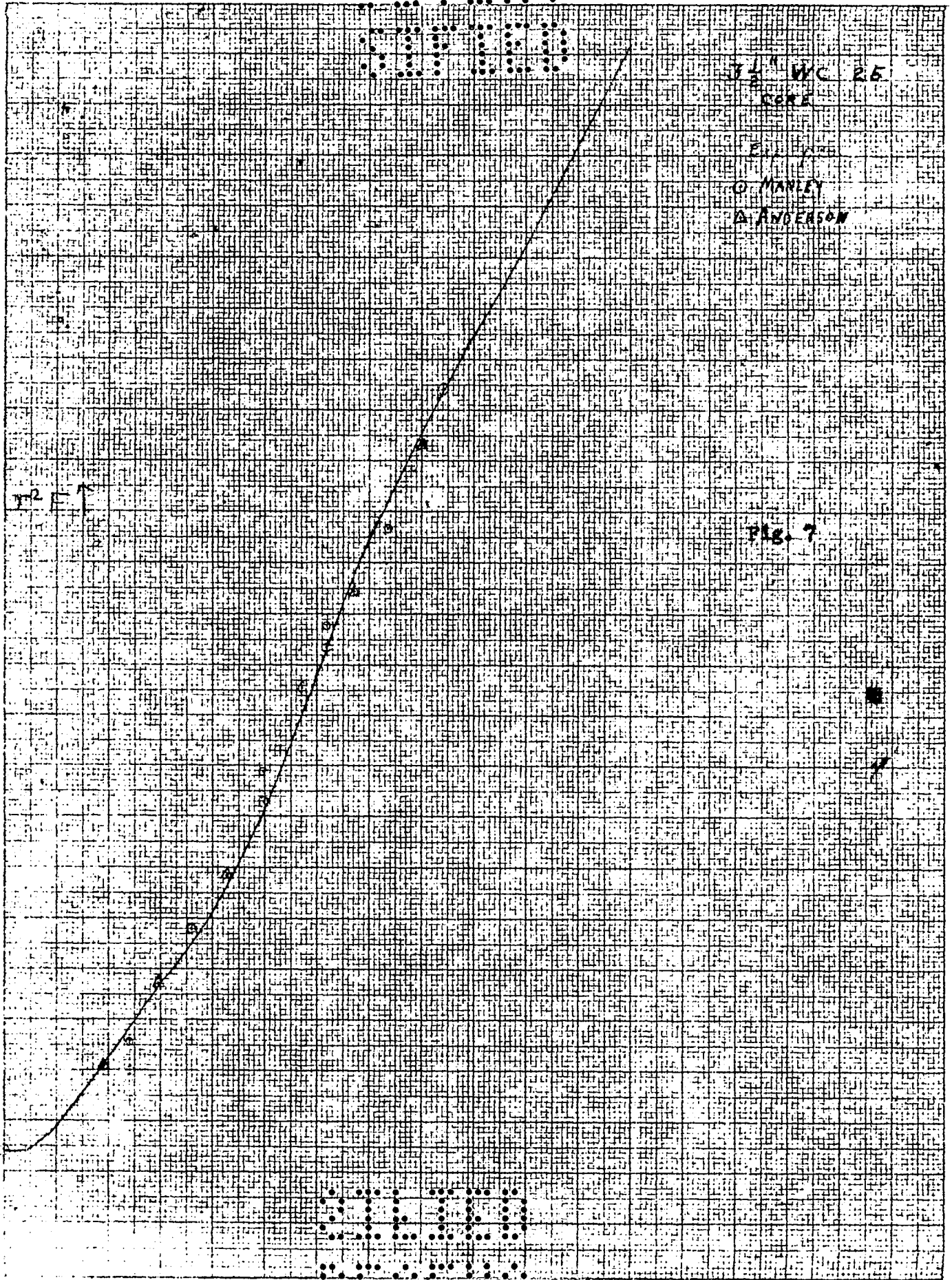
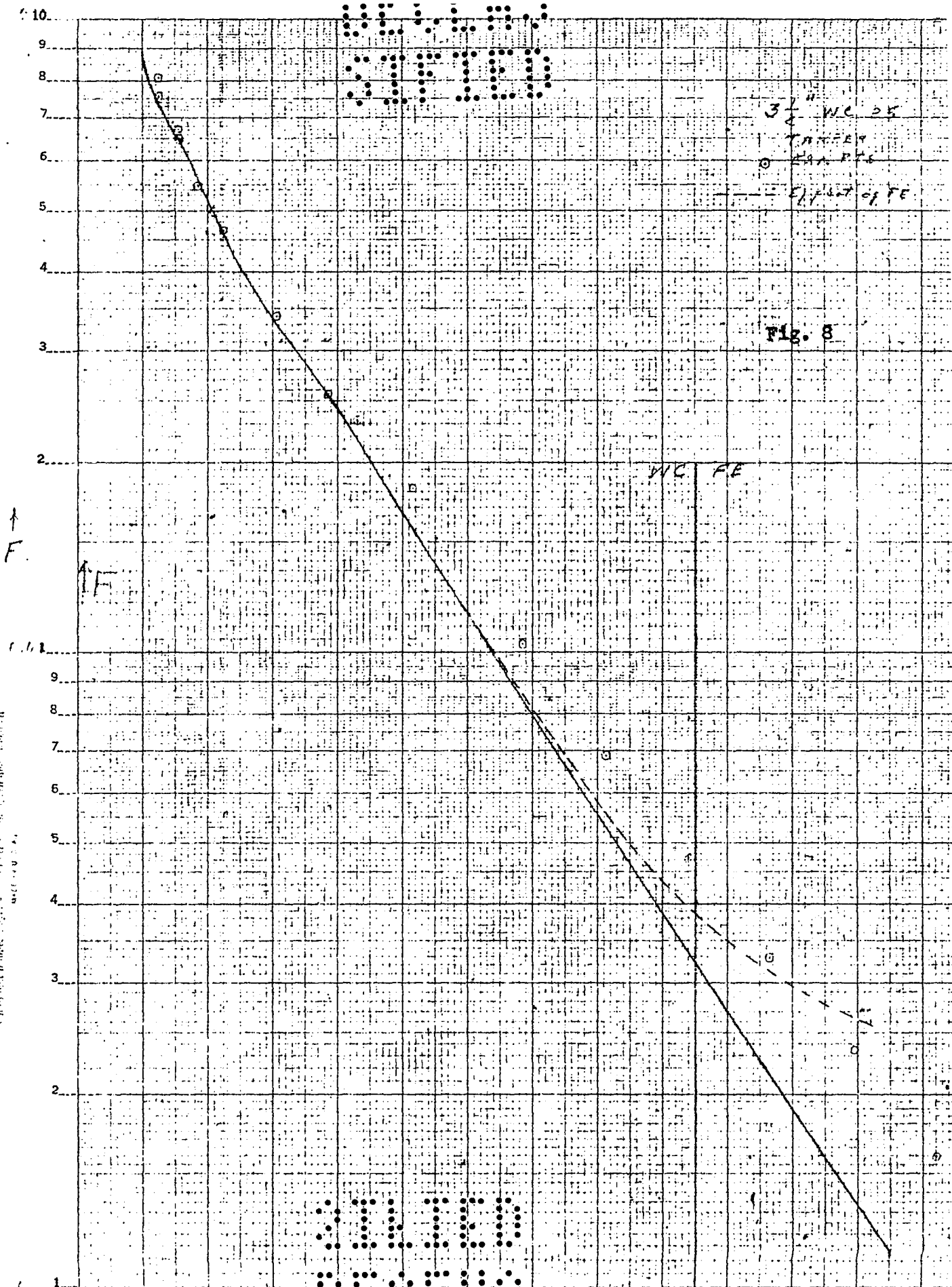


Fig. 7

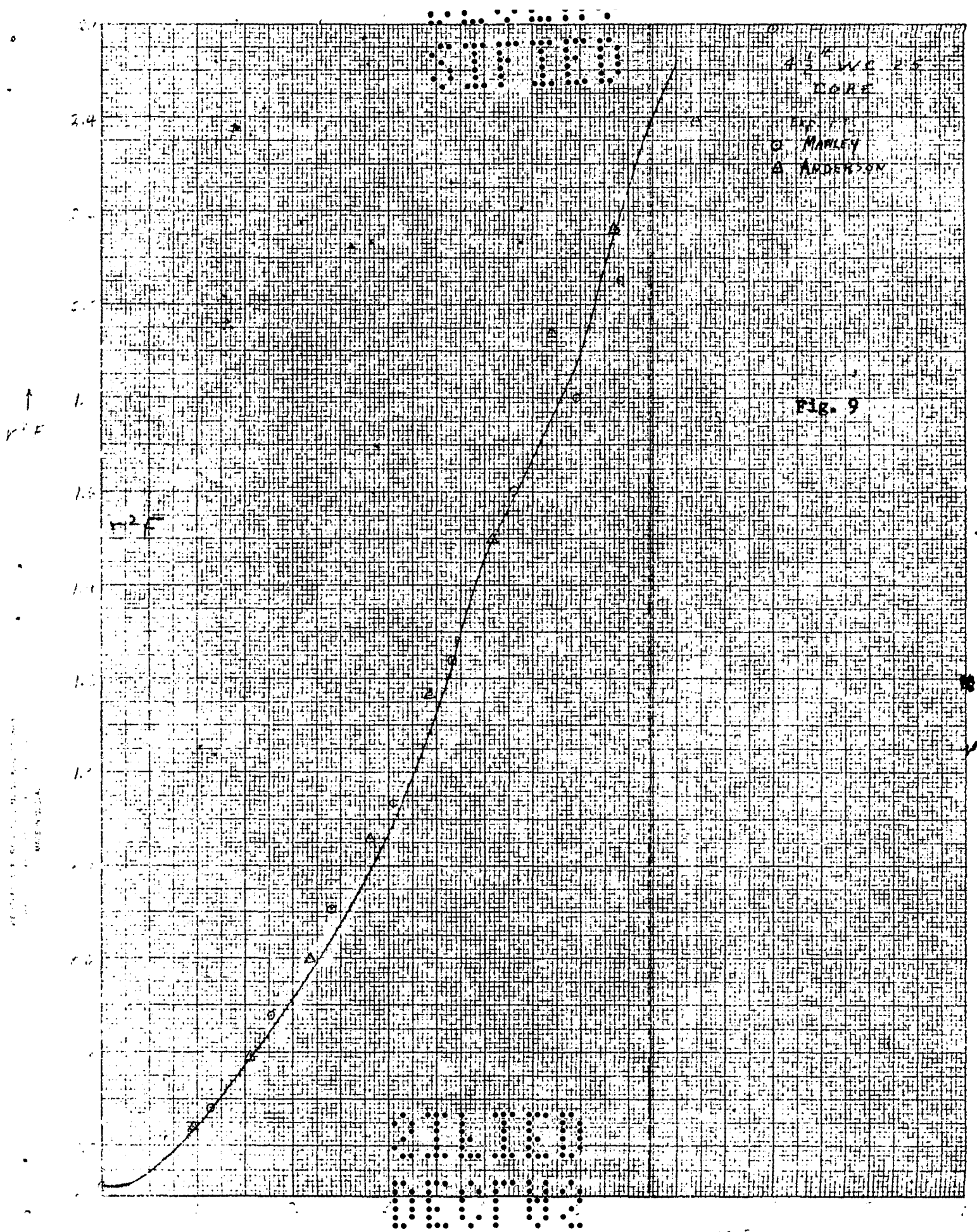
r²F

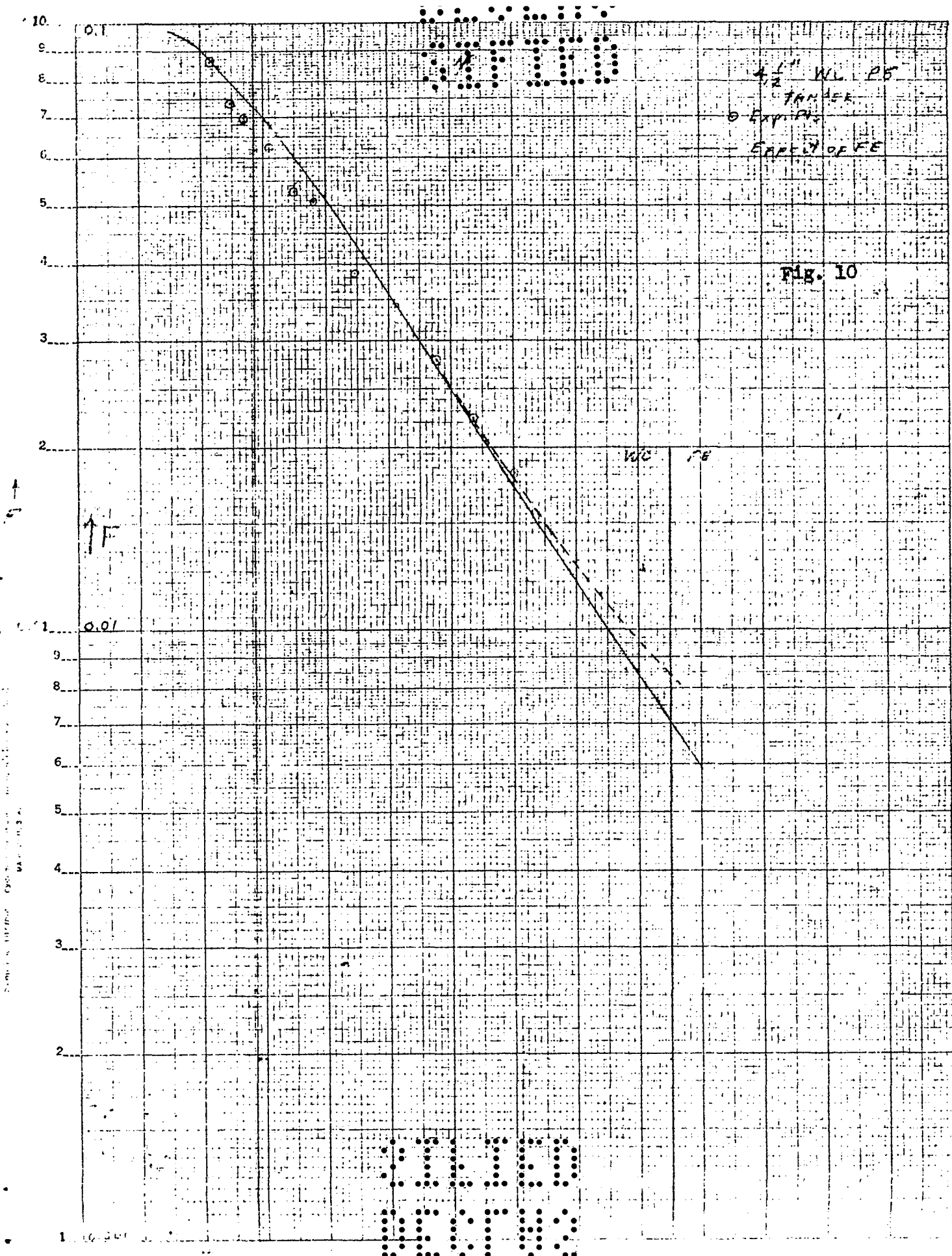
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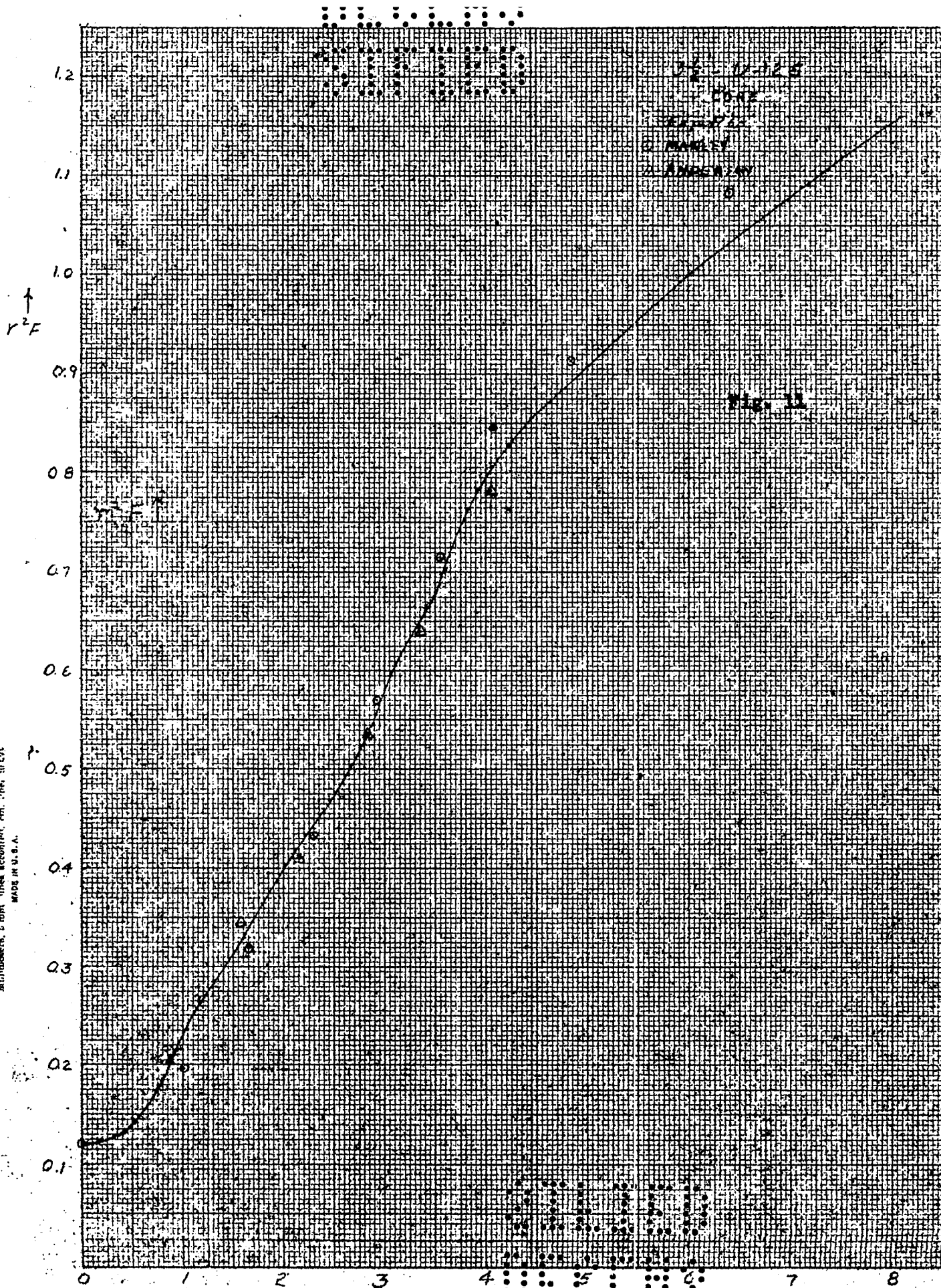
KUFFEL & ESSER CO. N.Y. N.Y. 10017
 500 10th Ave. Dept. 10017
 (212) 697-1001

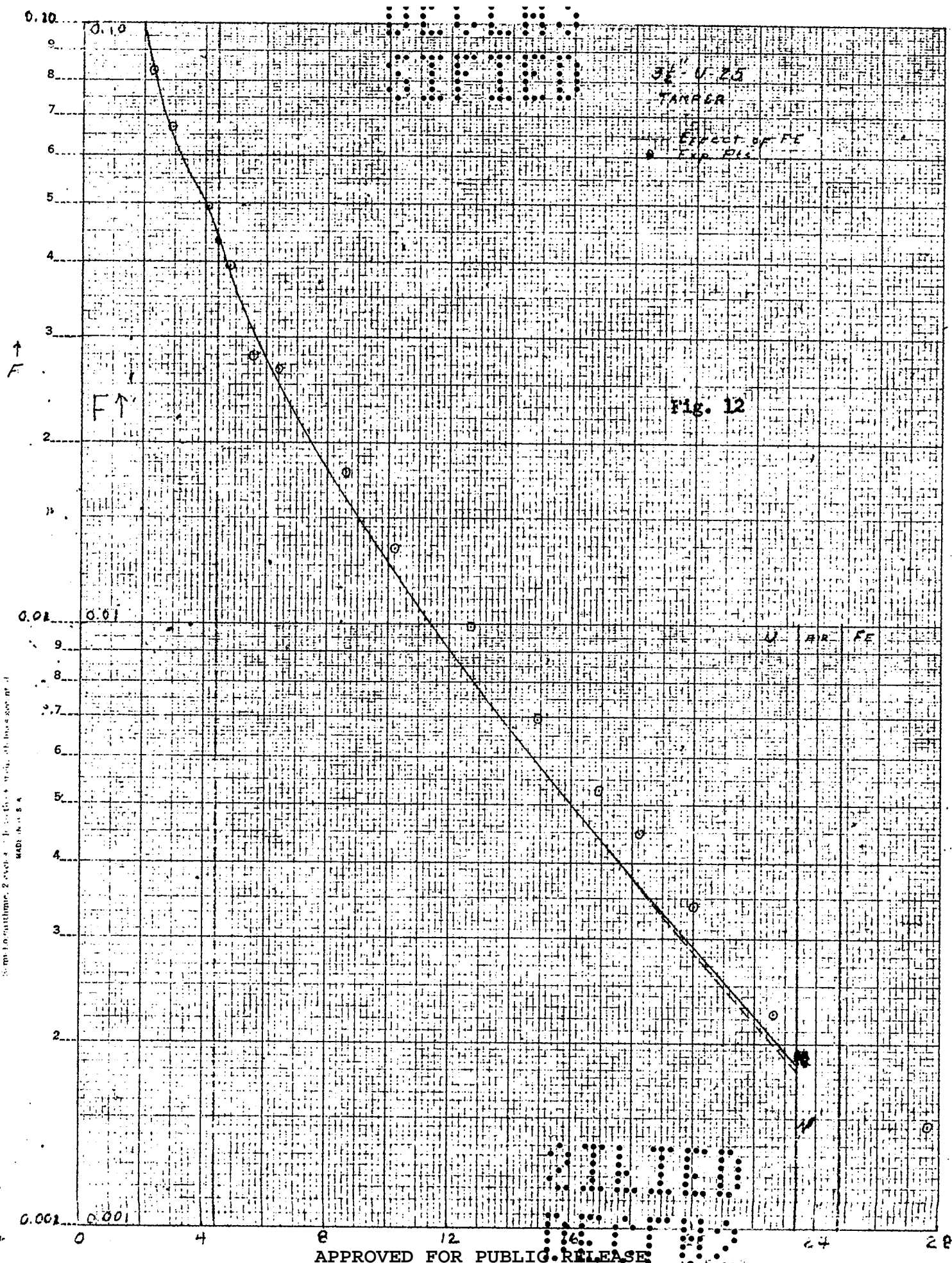




KEUFFEL & ESSER CO., N. Y. CO.
 Plotting Machine Co. 200 W. 4th St.
 New York 14, N. Y.

KEUFFEL & ESSER CO., N. Y. NO. 364H-14
Milwaukee, 5 mm. line spacing, cm. type trays
MADE IN U.S.A.

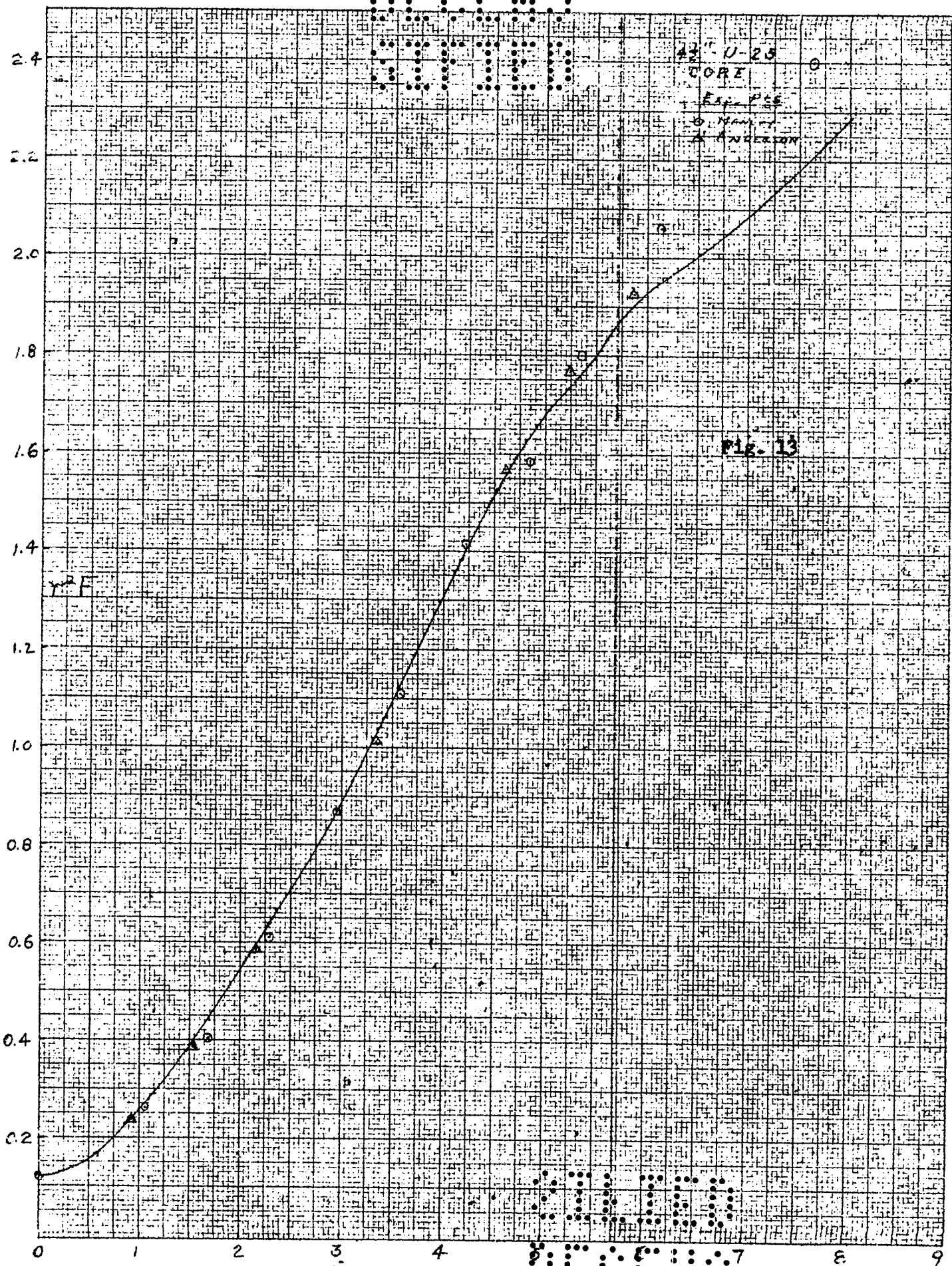




KEUFFEL & ESSER CO., N. Y. NO. 100-100
 35 mm. 10. Multiplier, 2.000 - 10.000
 MADE IN U.S.A.

$\gamma^2 F$

KEUFFEL & ESSER CO., N. Y. NO. 399-14
Millimeters, 5 mm. lines accented, cm. lines heavy,
MADE IN U. S. A.

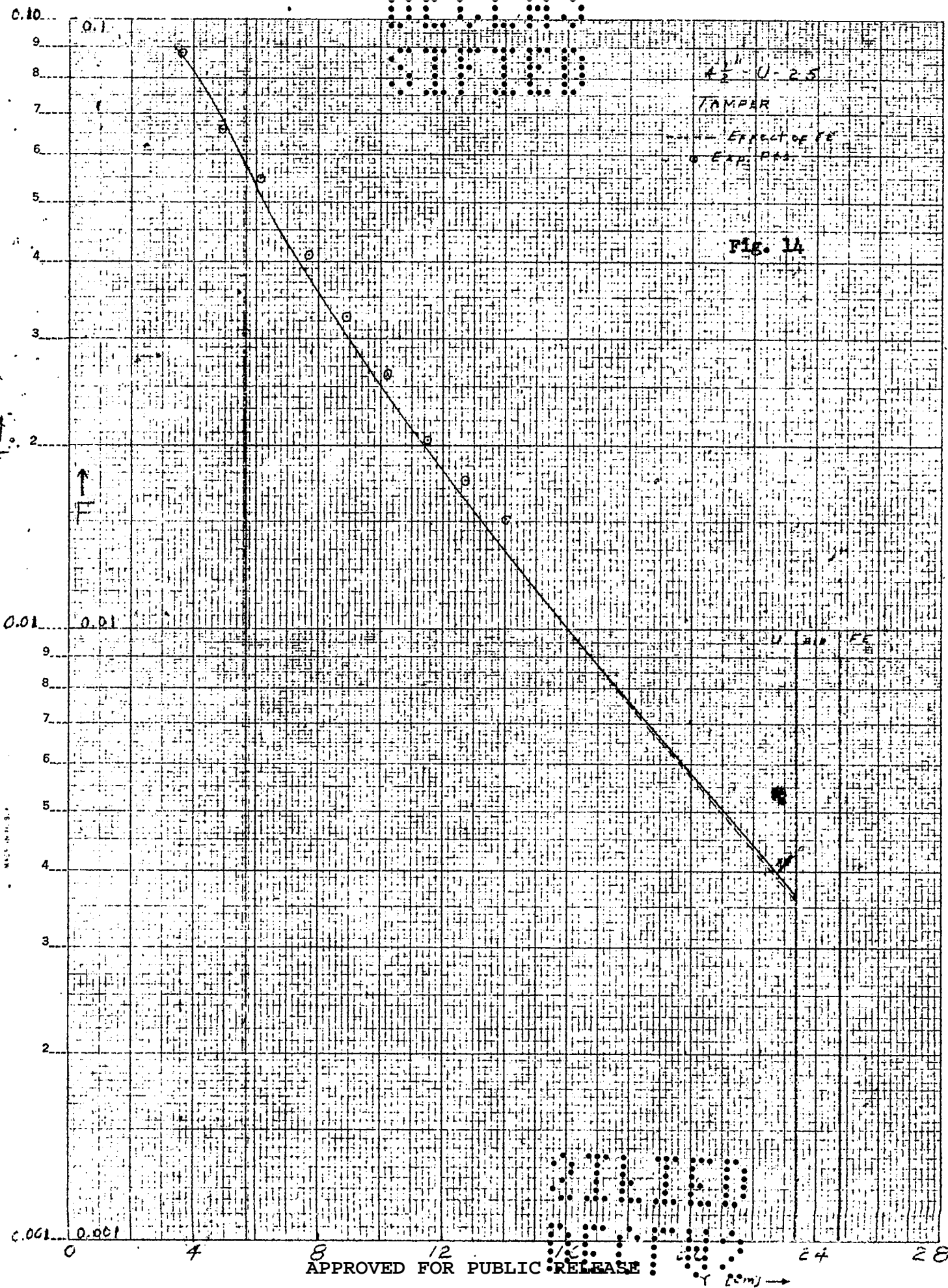


42" U-25
CORE
Exp. P-5
MANN
A. WILSON

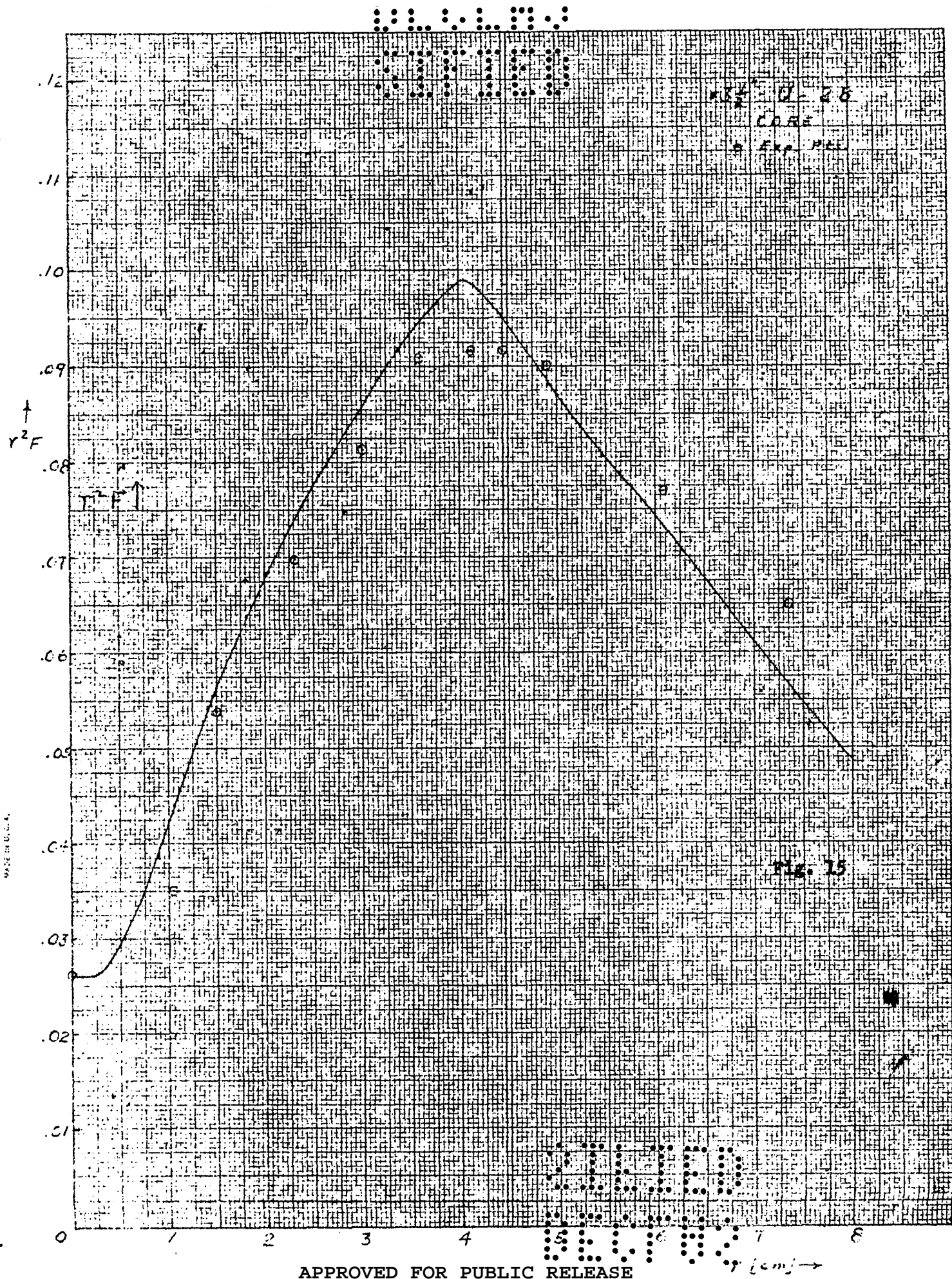
Fig. 13

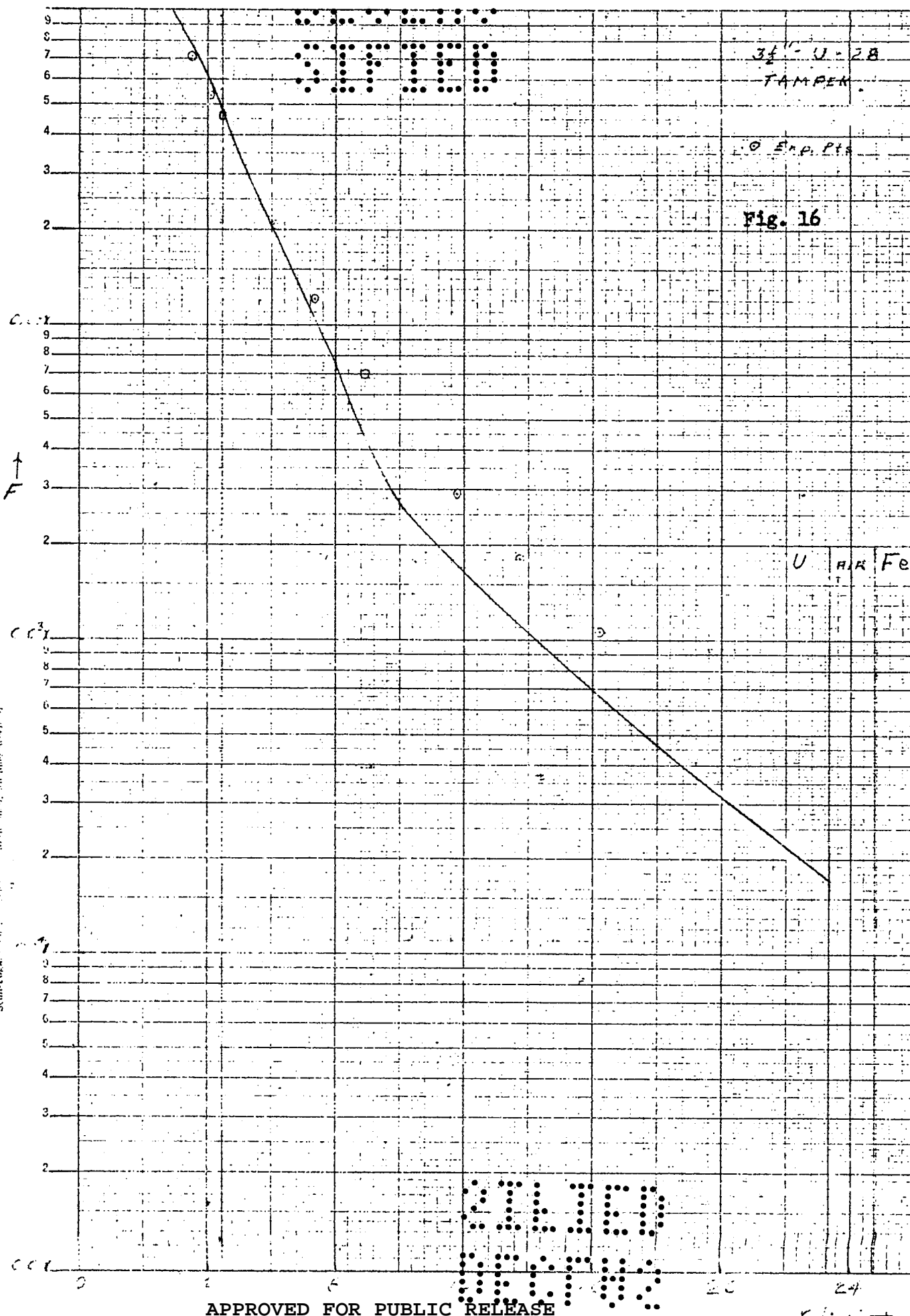
$\gamma^2 F$

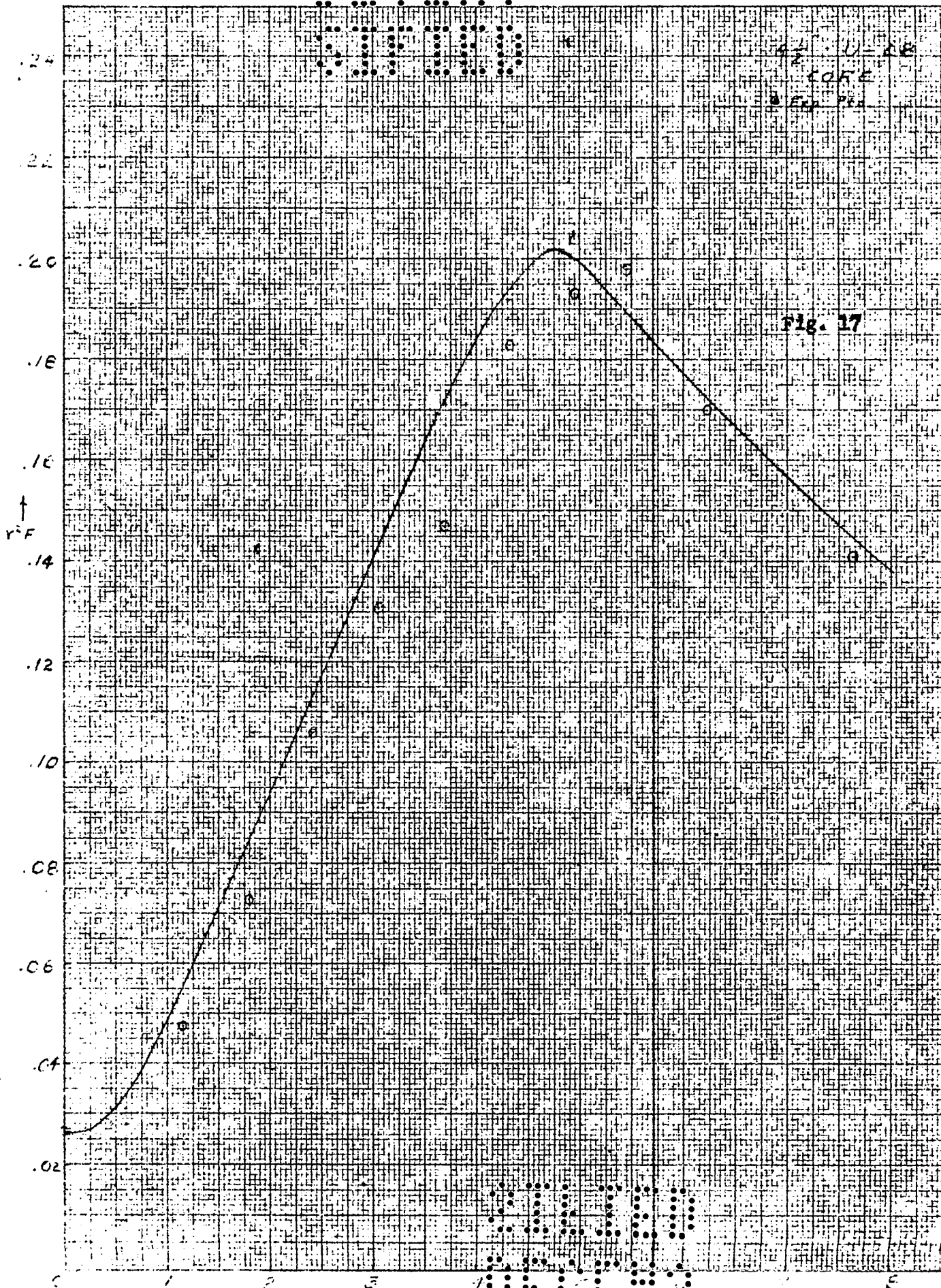
γ [cm] →



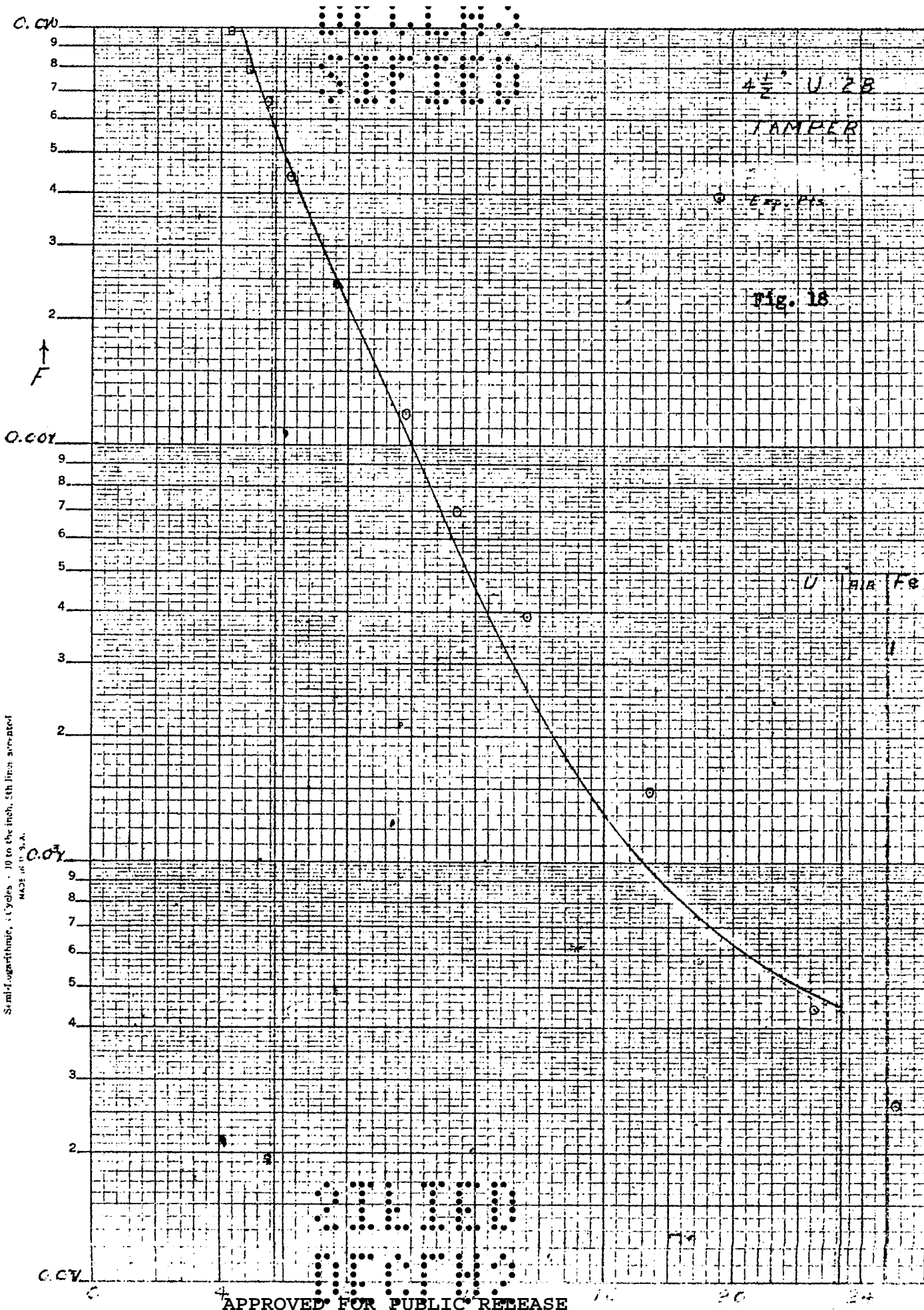
PERFFES A. ESSER CO. N. Y. NO. 0511
 Semi-logarithmic plot with logarithmic y-axis
 10/15/54







KEUFFEL & ESSER CO. INC. N.Y. N.O. 3-10-50
 MADE IN U.S.A.



KEUFFEL & ESSER CO., N. Y. NO. 359-71
 Semi-Logarithmic, Cycles, 10 to the inch, 15th inch standard
 MADE IN U.S.A.

SECRET

33 WVC 28
UOAK
Exp. P.

Fig. 19

↑
r²f

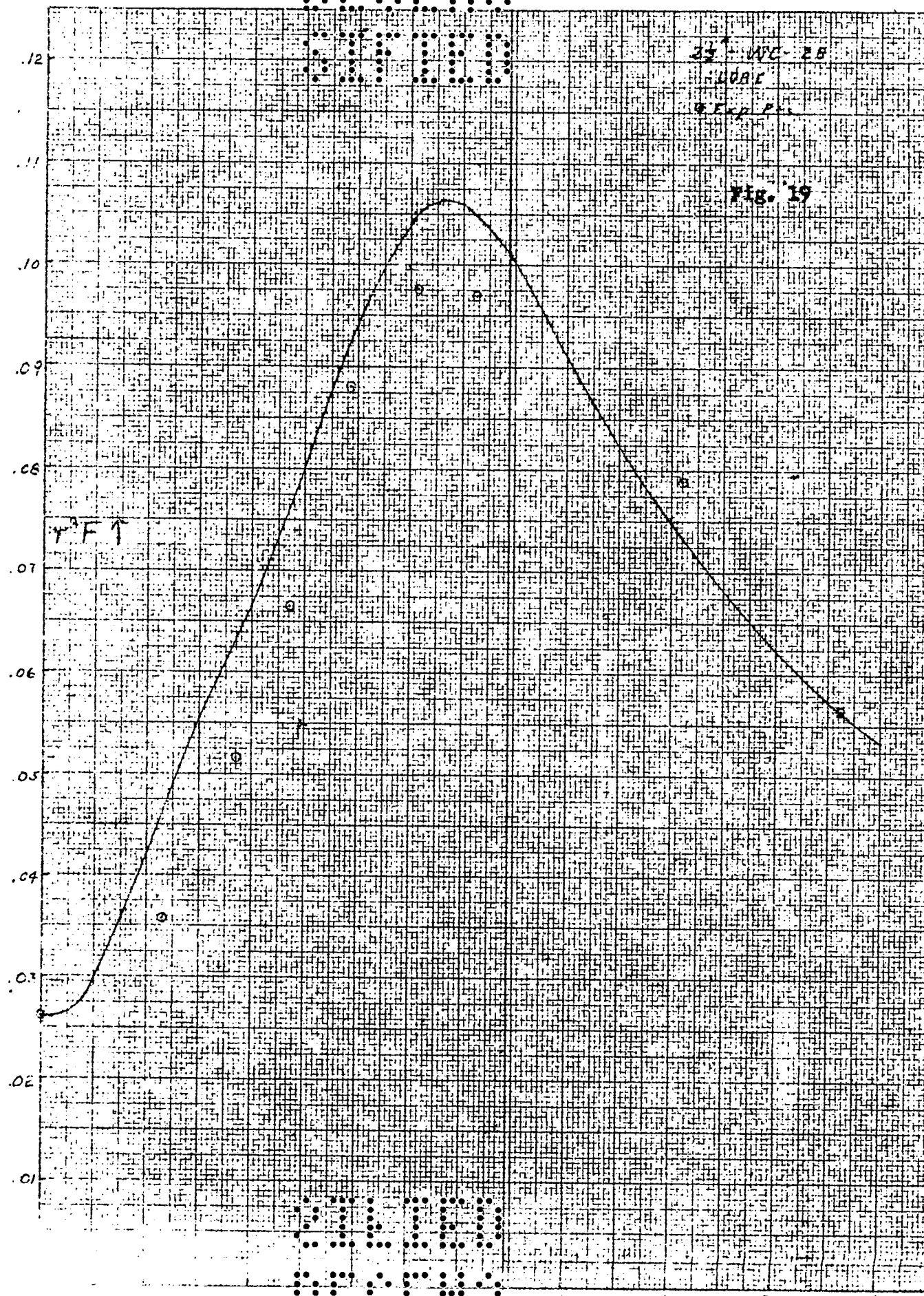
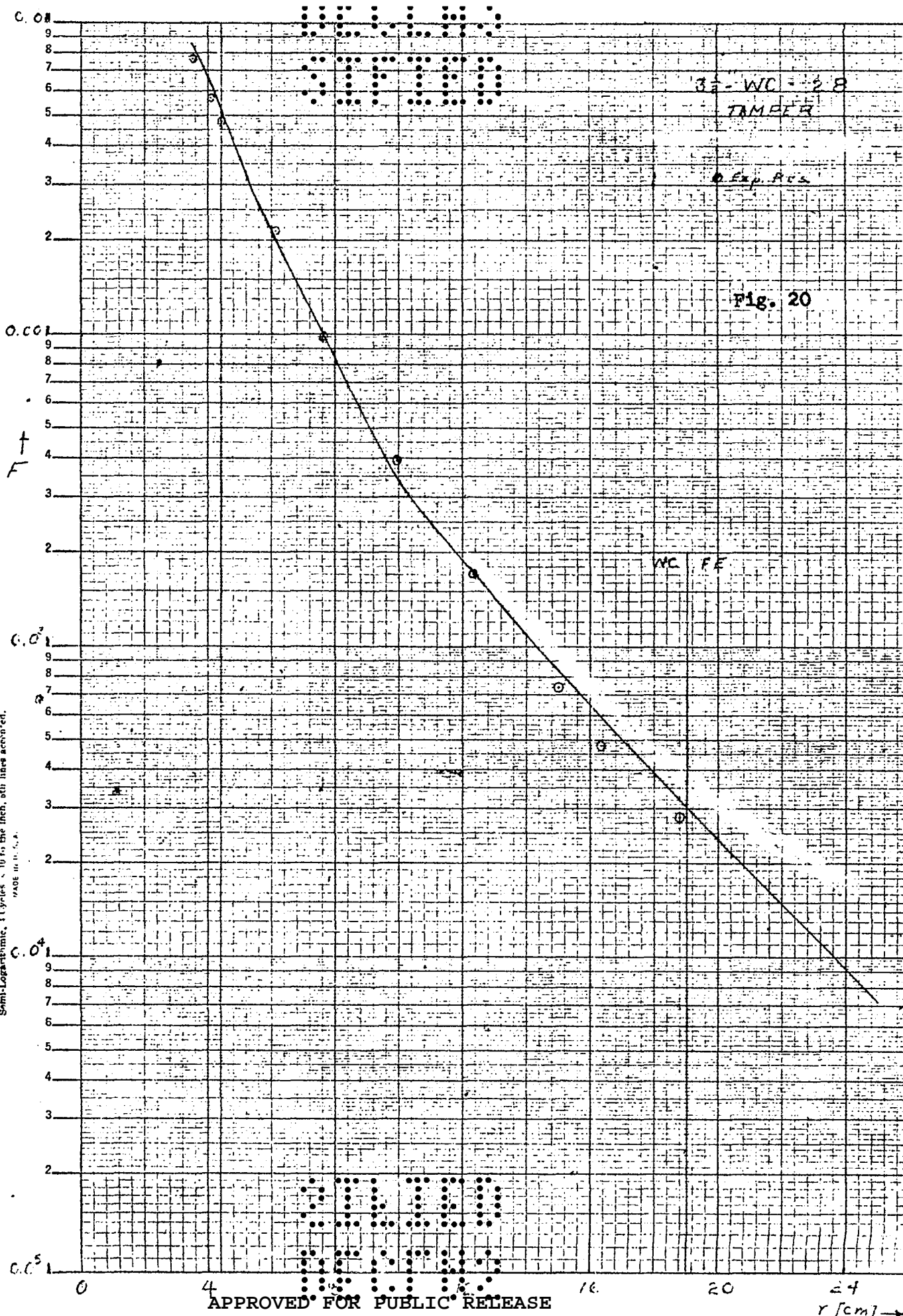


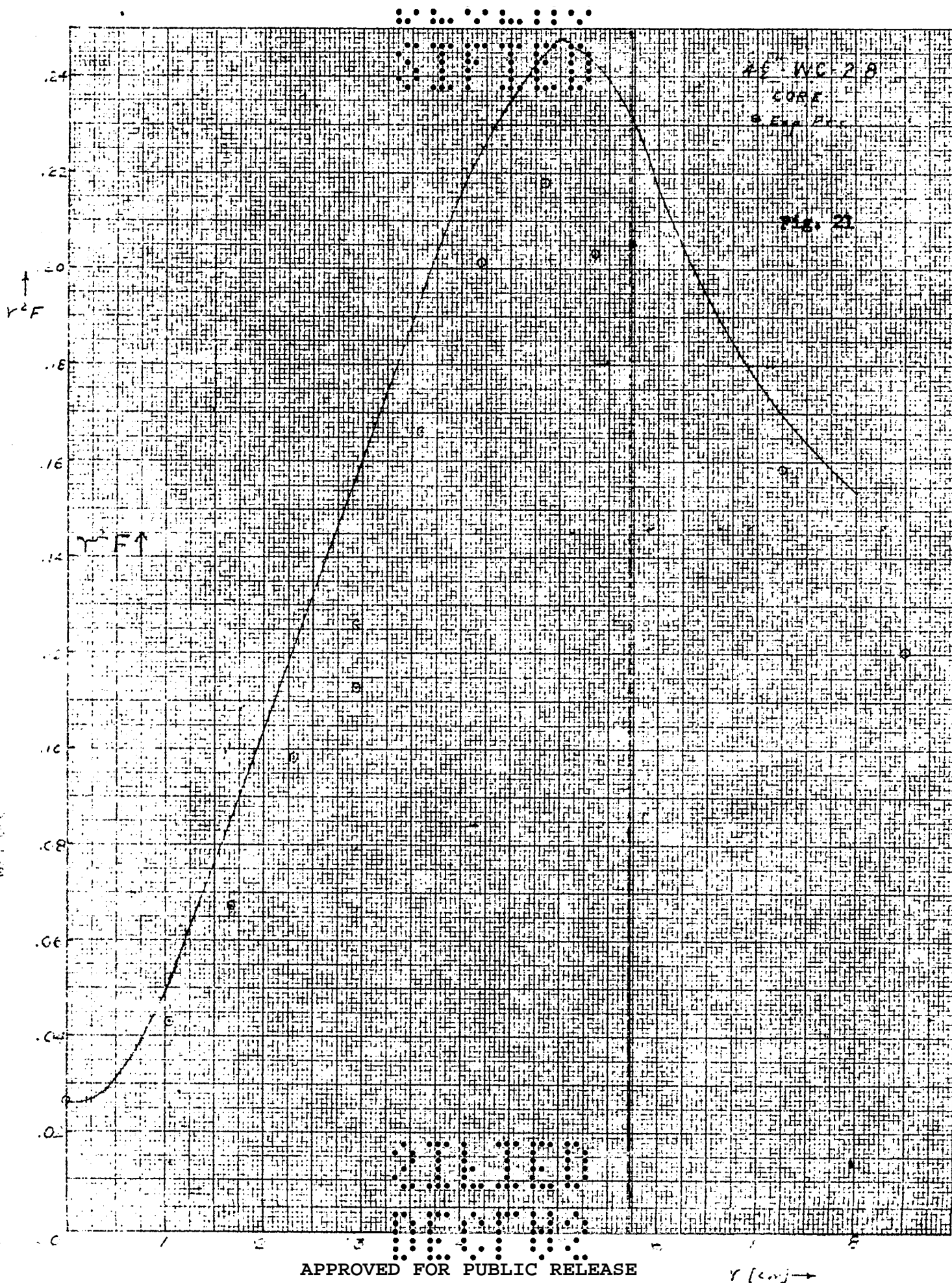
FIGURE 19 (continued) from page 18
MADISON, U.S.A.

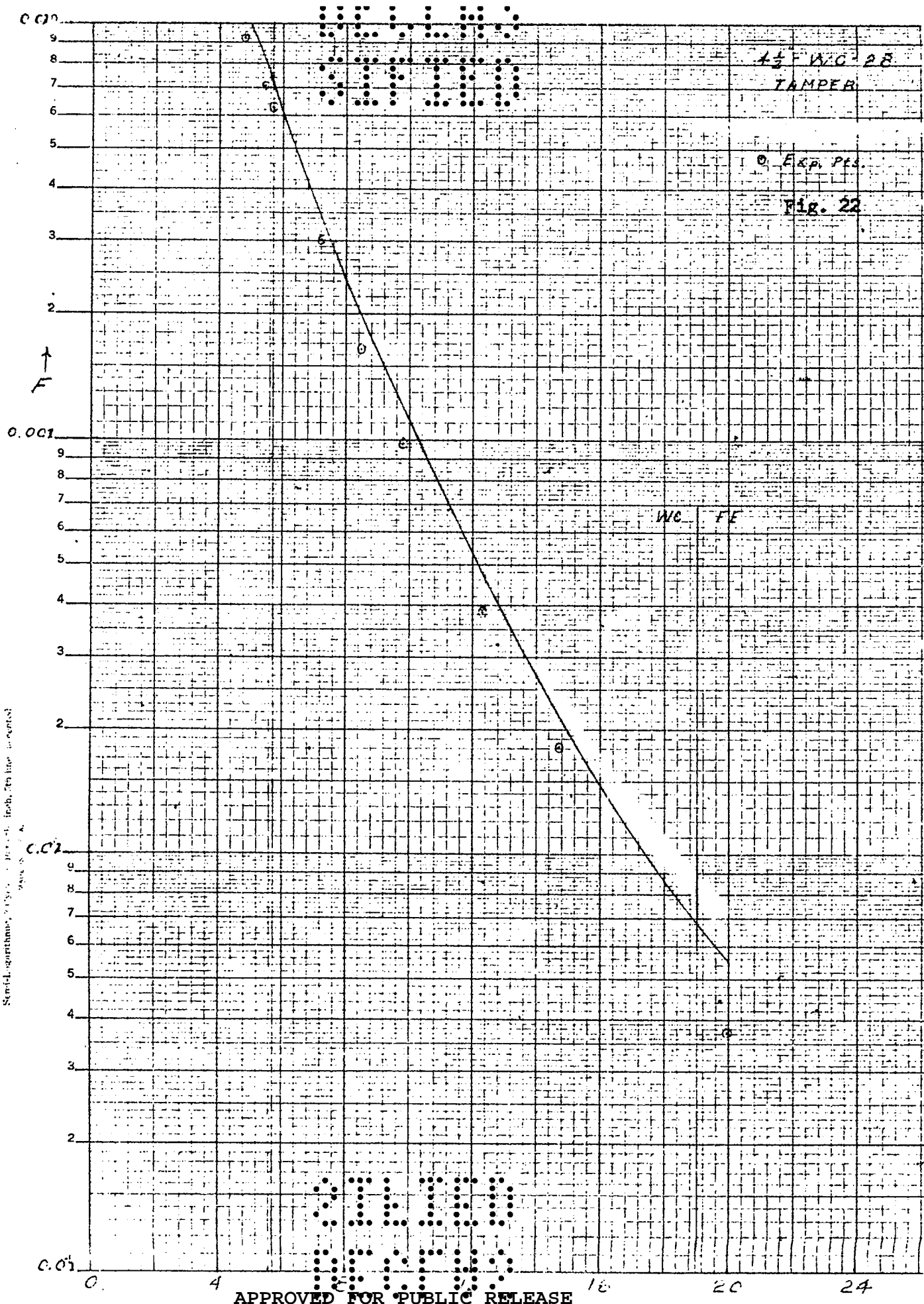
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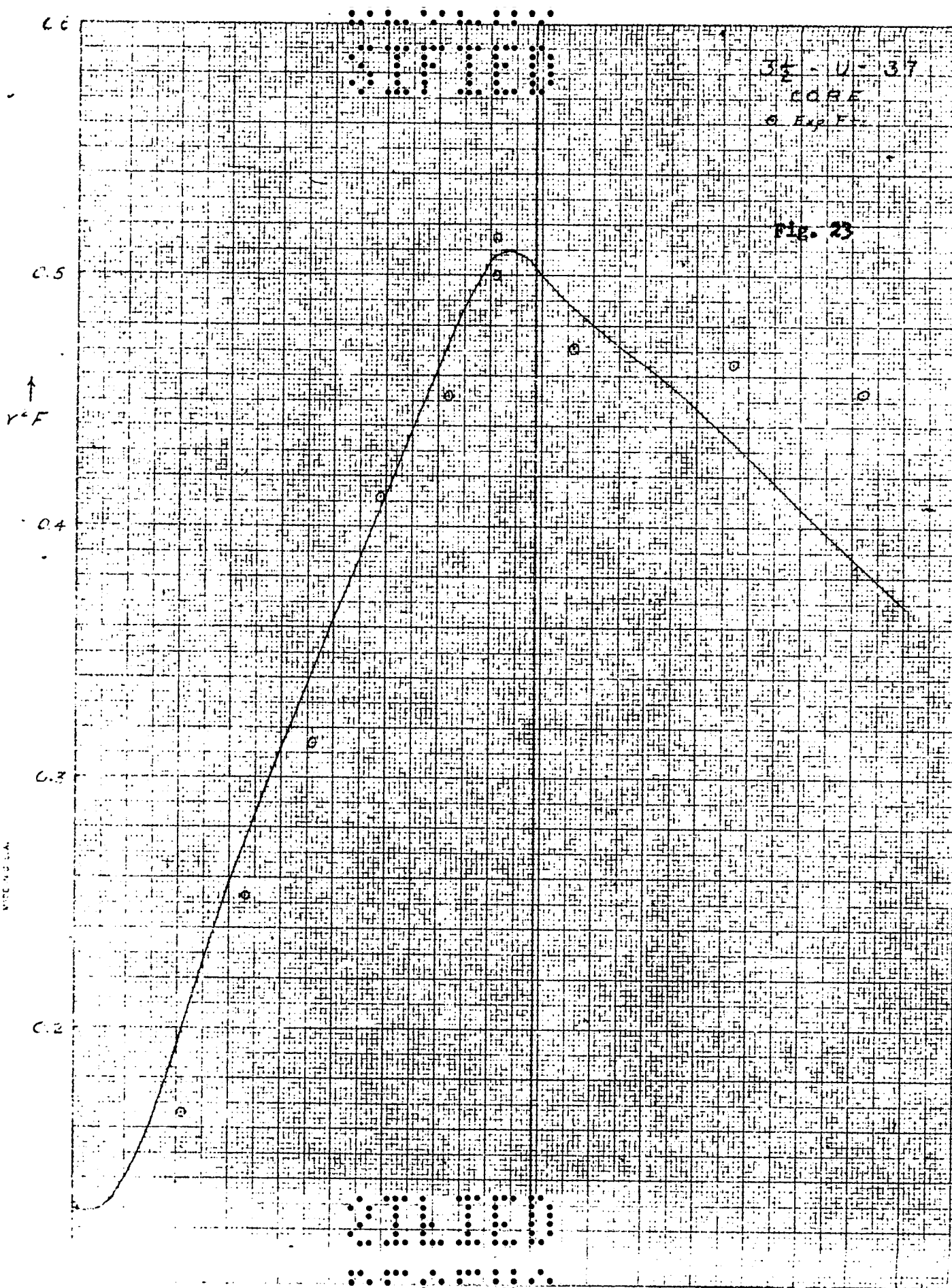
KEUFFEL & ESSER CO., N. Y. NO. 359-B1
Semi-Logarithmic, 1 Cycle, 10 to the inch, 5th lines accepted.
MADE IN U.S.A.

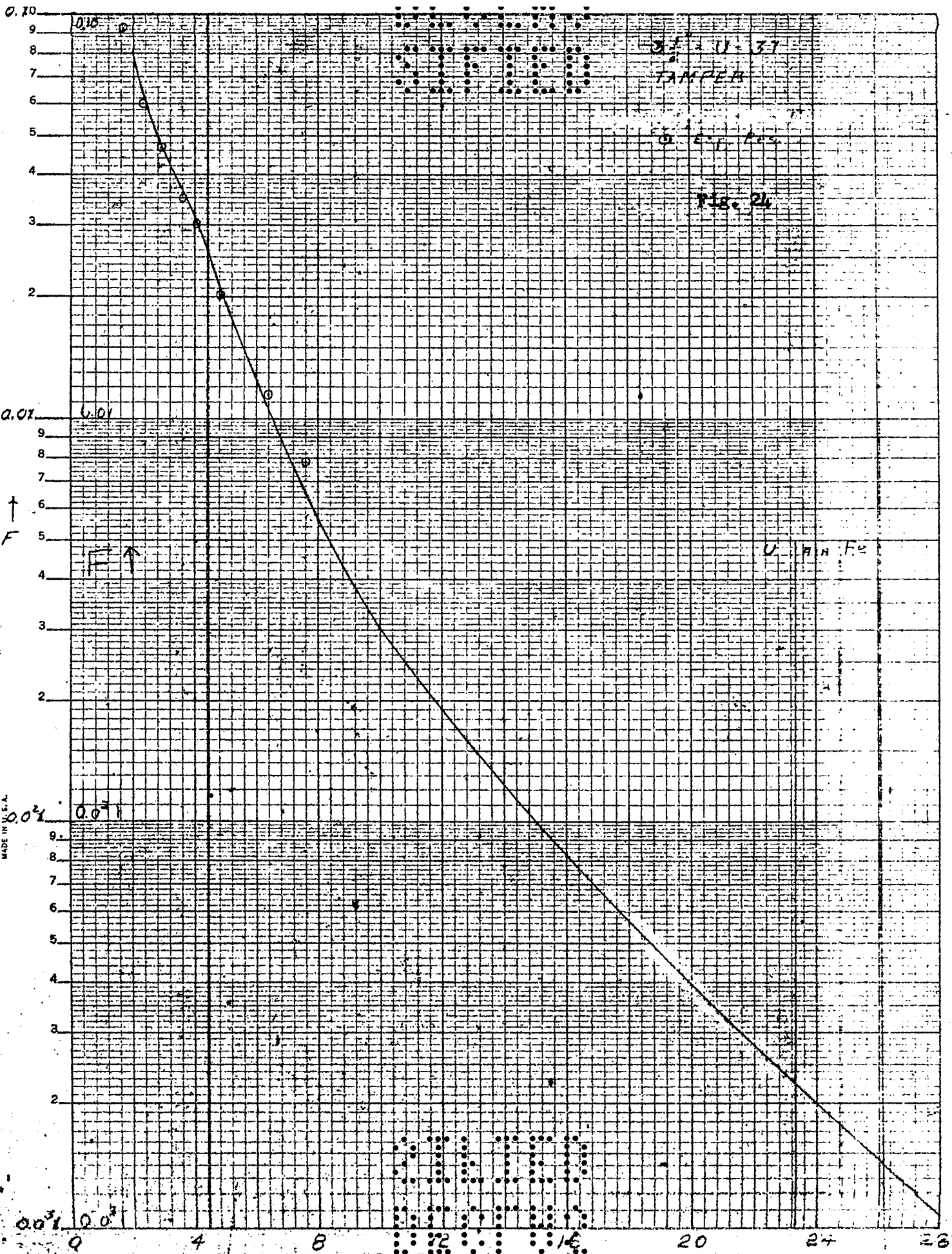
Fig. 20



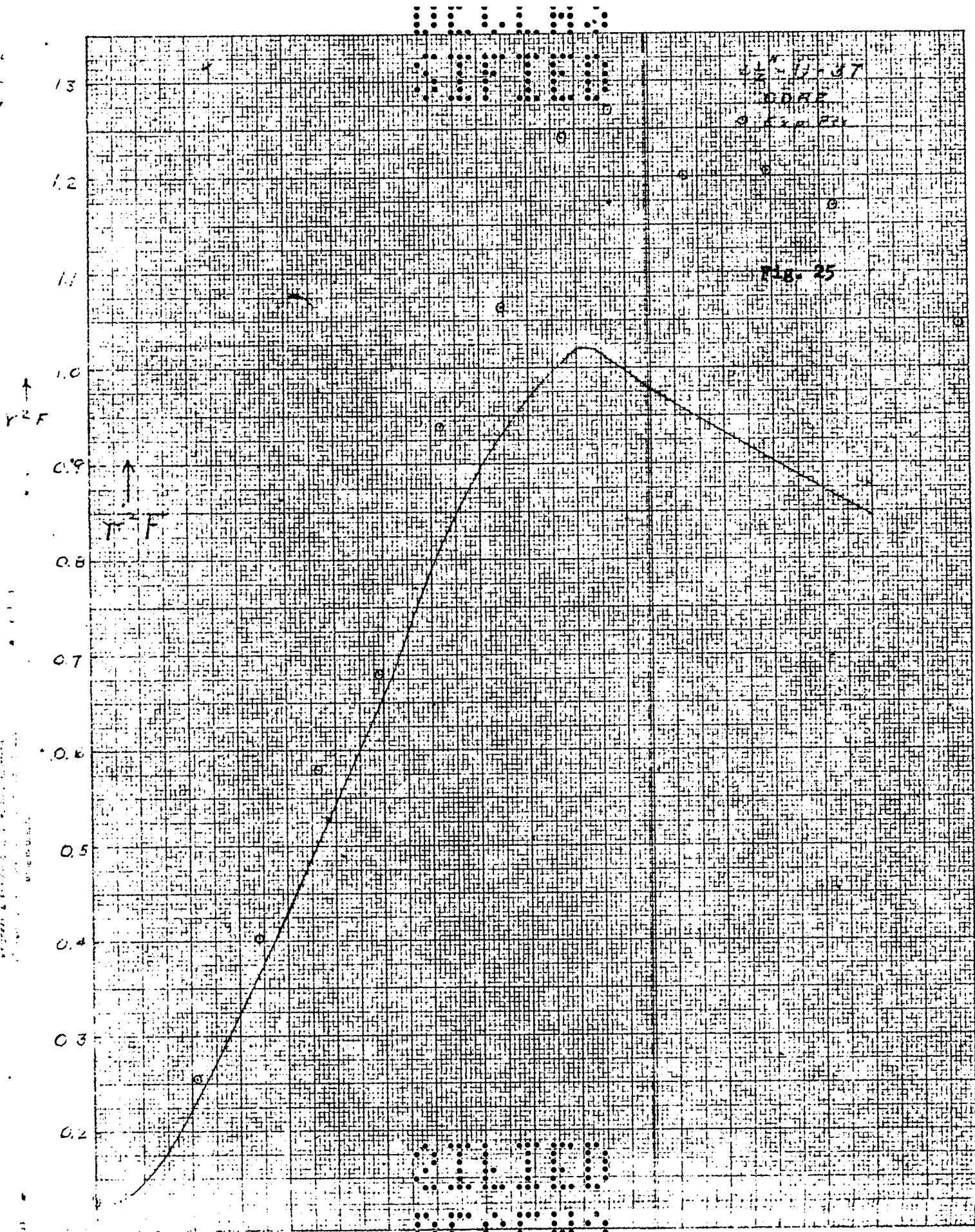


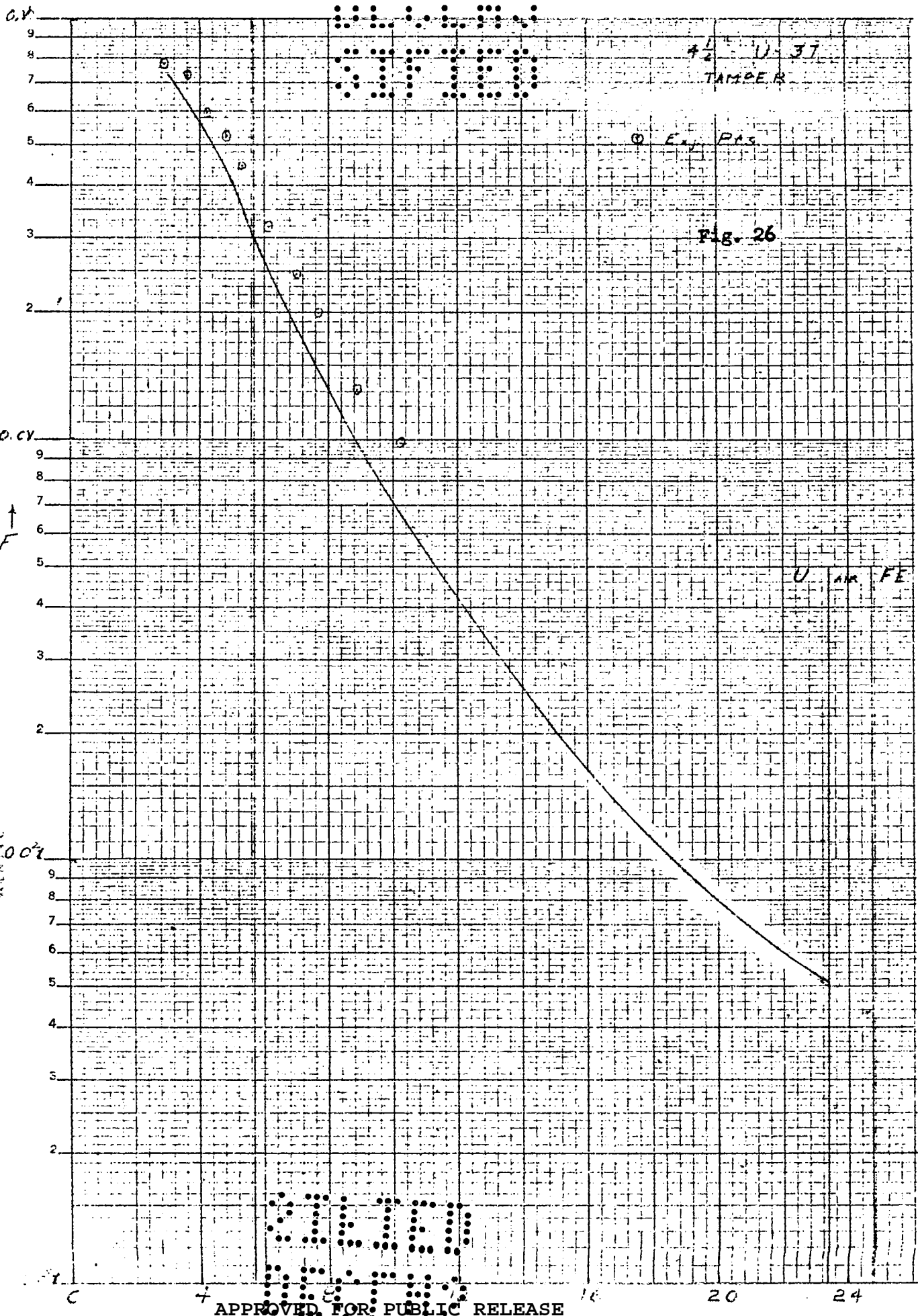
KEUFFER & ESSER CO., N. Y. NO. 37-571
 Semi-Logarithmic Plotting Machine, 20 inch, 20 line, mounted
 MAR. 20, 1944





KEUFFEL & ESSER CO., N. Y. NO. 380-71
Semi-Logarithmic, 3 Cycles x 10 to the inch, 5th lines accentuated
MADE IN U.S.A.

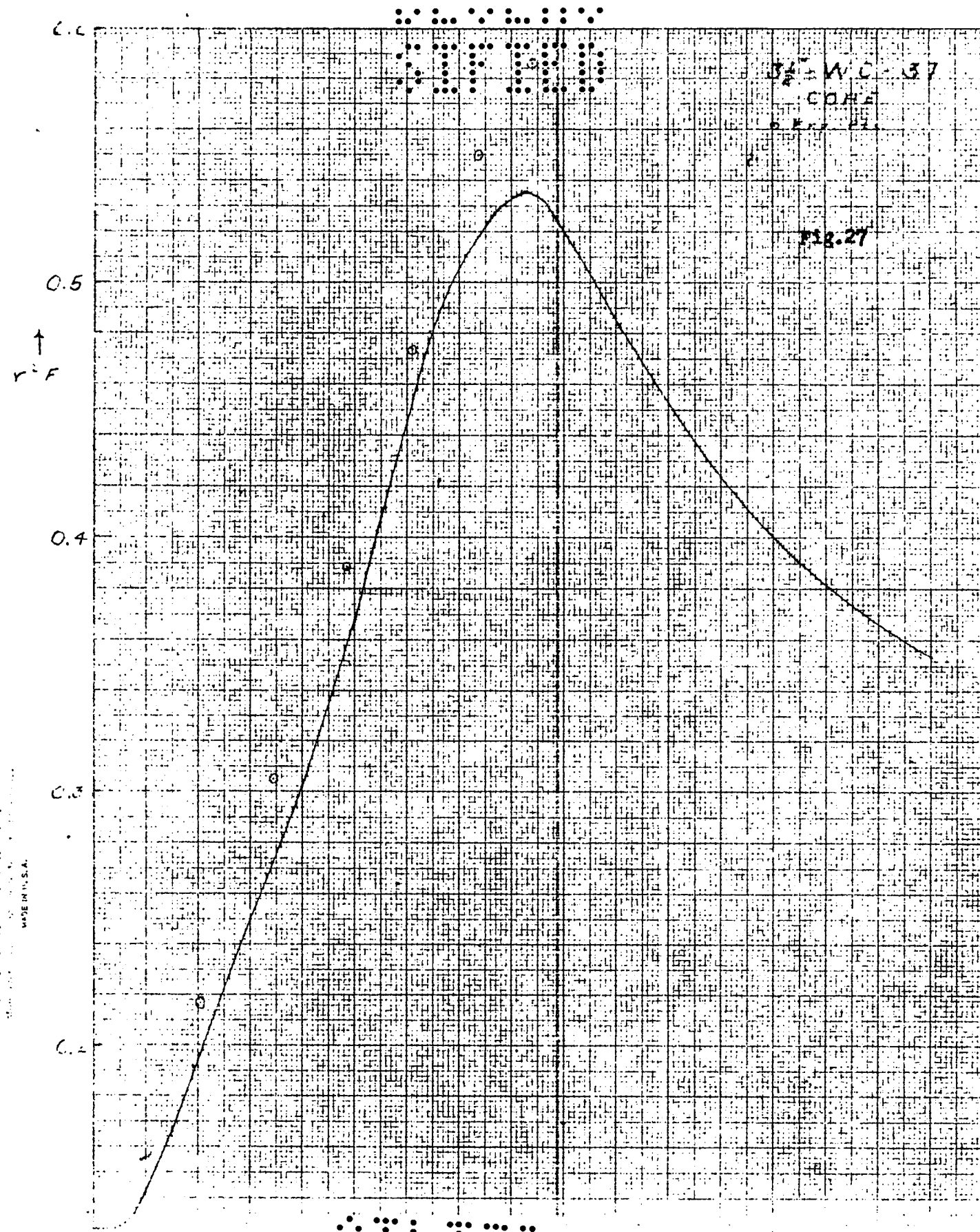




KEUFFEL & ESSER CO. N. Y. NO. 35-71
 Printing on microfilm. Cycle 1. 100% reproduction.

100%

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KEUFFEL & ESSER CO., N. Y. NO. 359-81
Semi-Logarithmic, 1 Cycle X 10 to the inch, 5th lines arc-cent.
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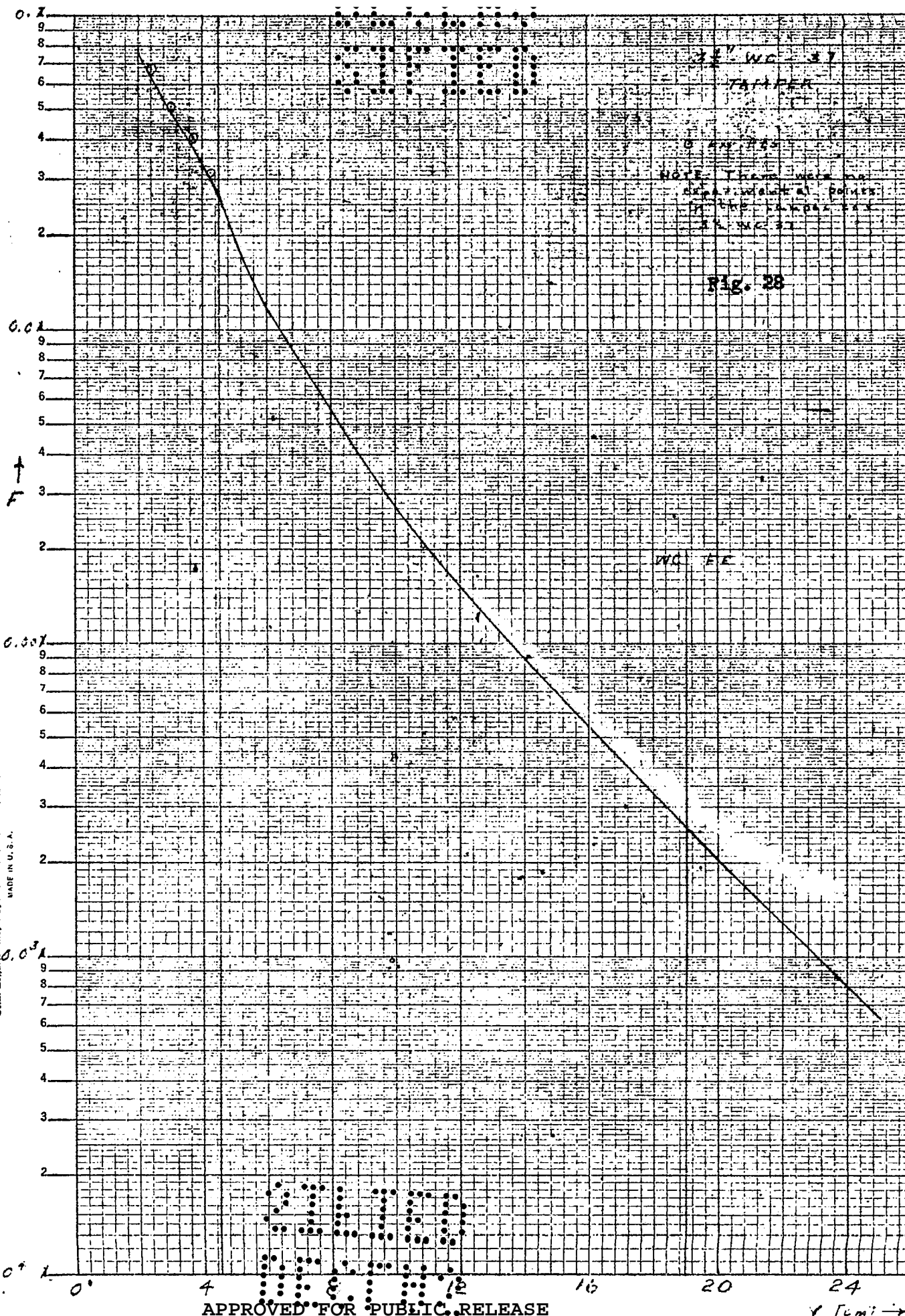
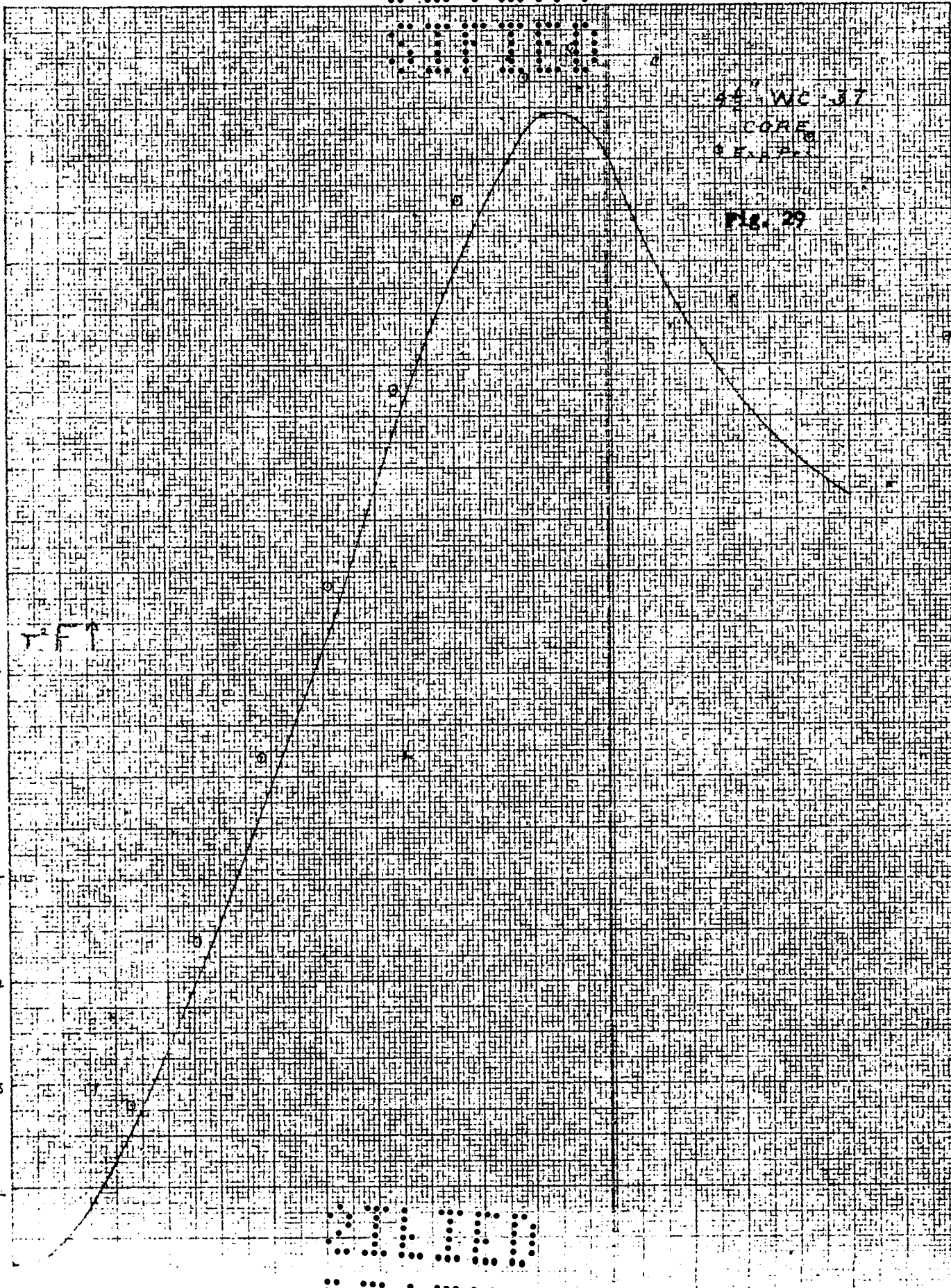


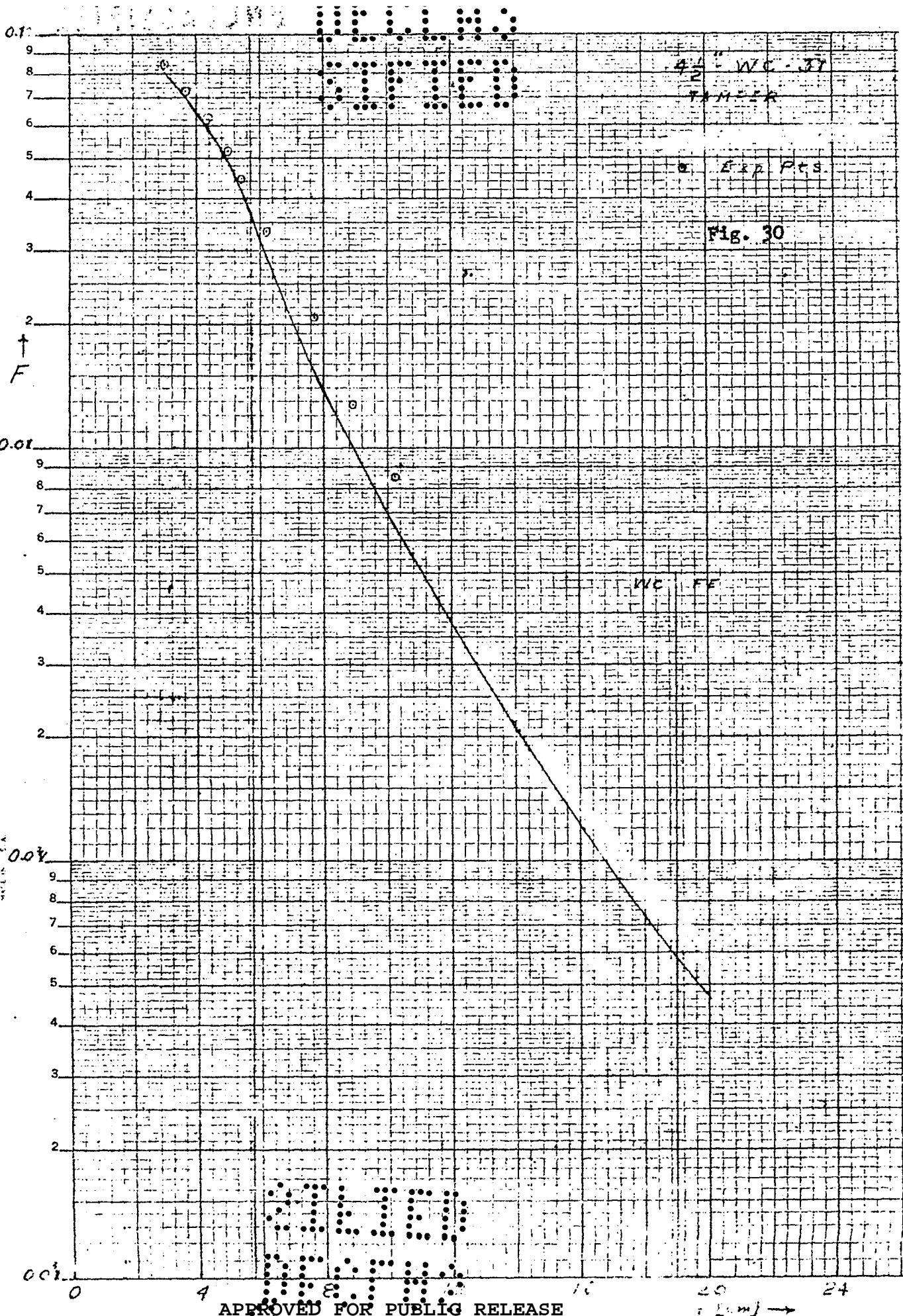
Fig. 28

r [cm] →

↑
Y²F



FOR THE ESTABLISHMENT OF THE NATIONAL BUREAU OF STANDARDS
MADE IN U.S.A.



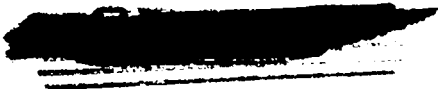
KEUFEL & ESSER CO., N. Y. N.C. 30071
 Semi-Logarithmic Plotting Paper
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